Properties of BA networks

Introduction to Network Science

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Contents

- Degree distribution under the BA model
- Distance distribution under the BA model
- Clustering coefficient under the BA model

BA model means Barabási-Albert model (preferential attachment)

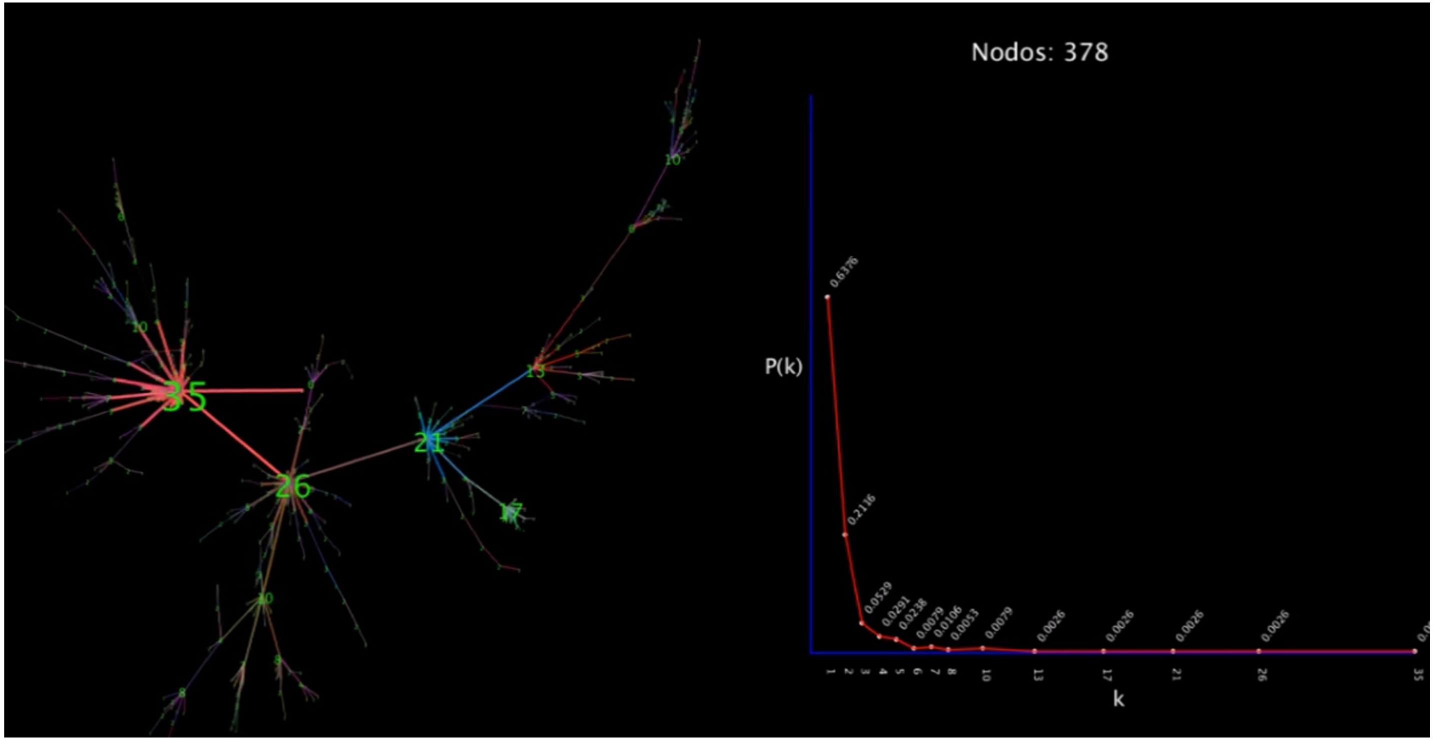
The Barabási-Albert (BA) model

- Network starts with m_0 nodes connected arbitrarily as long as their degree is ≥ 1
- At every time step we add 1 node
- This node will have $m \leq m_0$ outlinks
- The probability of an existing node of degree k_i to gain one such link is

$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

In an ER network, $\Pi(k_i) = \dfrac{1}{N-1}$

Video of degree distribution



https://www.youtube.com/watch?v=5RIQweqPT6A

Degree $k_i(t)$ as a function of time

$$\frac{d}{dt}k_i = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

$$\sum_{j=1}^{N-1} k_j = L(0) + 2m(t-1) \approx 2m(t-1)$$

(For large t)

$$\frac{d}{dt}k_i = \frac{mk_i}{2m(t-1)} = \frac{k_i}{2t-2} \approx \frac{k_i}{2t}$$

Degree $k_i(t)$ ... continued

$$\frac{d}{dt}k_i(t) = \frac{k_i(t)}{2t}$$

$$\frac{1}{k_i(t)}\frac{d}{dt}k_i(t) = \frac{1}{2t}$$

$$\int_{t=t_i}^{t}\frac{1}{k_i(t)}\frac{d}{dt}k_i(t)dt = \int_{t=t_i}^{t}\frac{1}{2t}dt$$

(t_i is the creation time of node i)

$$\log k_i(t) - \log k_i(t_i) = \frac{1}{2}\log t - \frac{1}{2}\log t_i$$

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

Note: in exams for this course, you will **not** be asked to solve differential equations on your own

Degree $k_i(t)$ ... continued

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

Is the degree growth linear, super-linear, or sub-linear? Intuitively, why?

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}} = m\left(\frac{t}{t_i}\right)^{\beta}$$

$\beta = 1/2$ is called the dynamical exponent

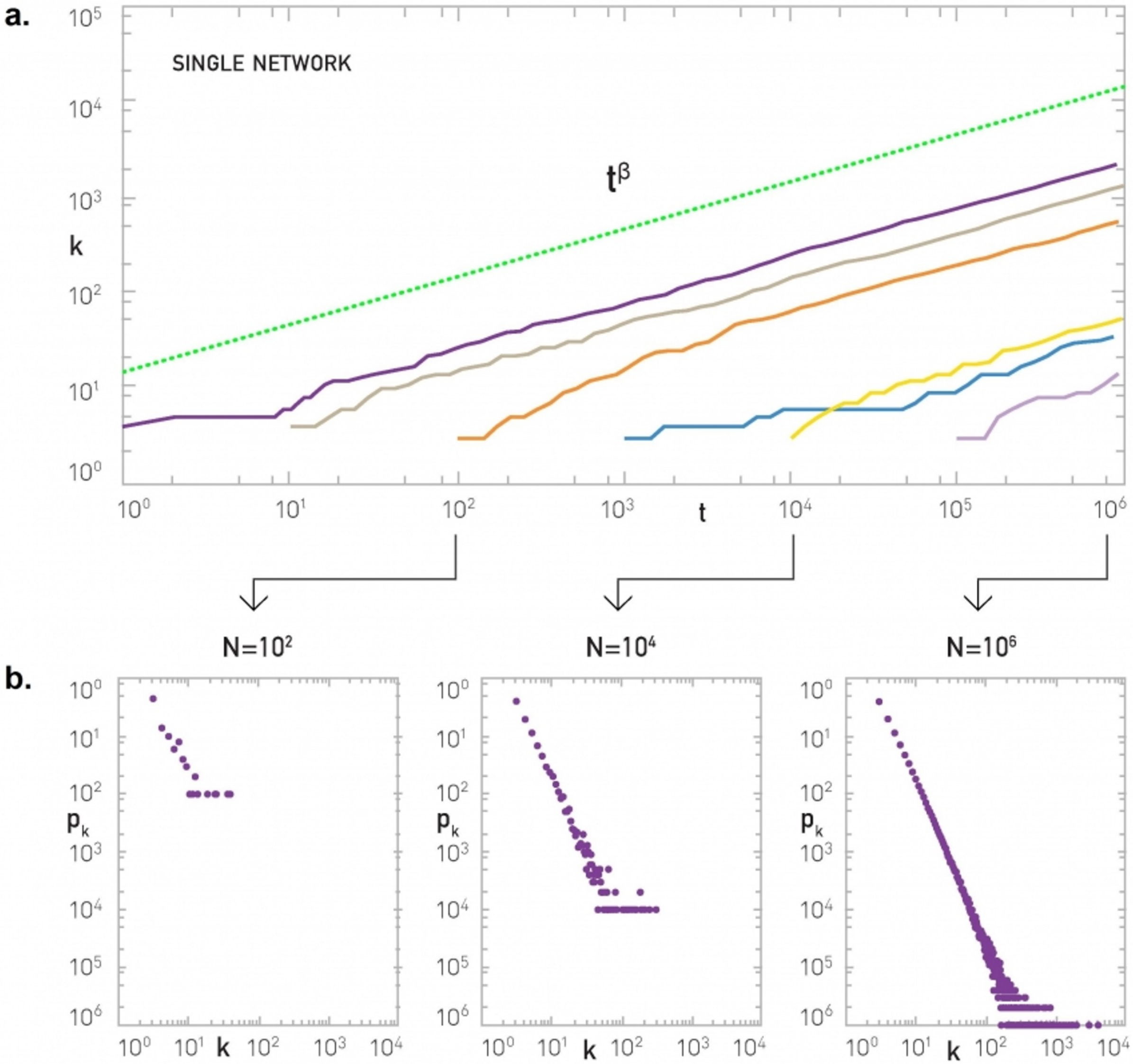
Degree $k_i(t)$ … consequences

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t} = \frac{m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}}{2t} = \frac{m}{2\left(t \cdot t_i\right)^{\frac{1}{2}}}$$

If $t_i < t_j$ (node *i* is older than node *j*), what do we expect of k_i and k_j?



Simulation results

Model

Nodes with
t_i = 1, 10, 100, 1000, 10000, ...

Degree distribution

- The distribution of the degree follows

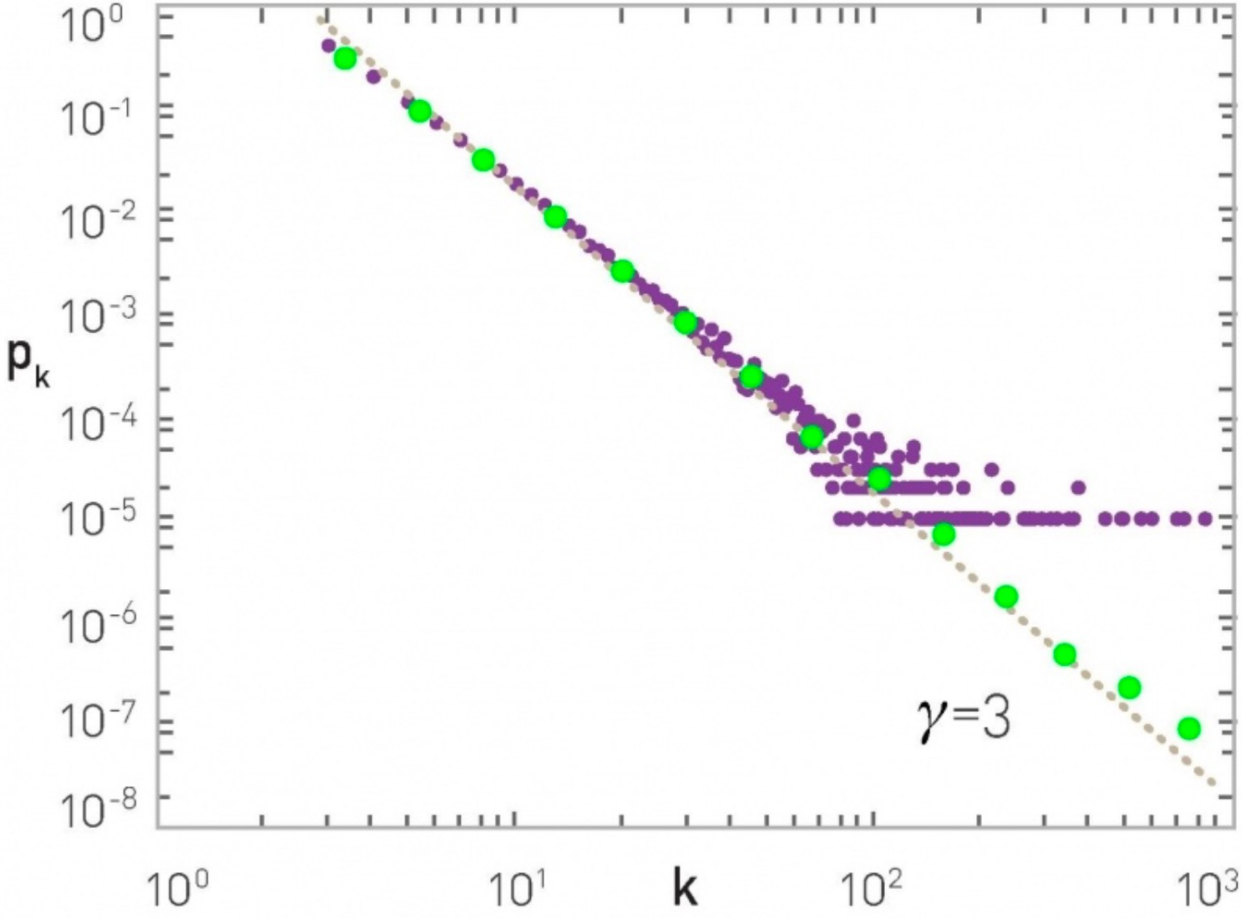
$$p(k) \approx 2m^2/k^3$$

- (Proof later)

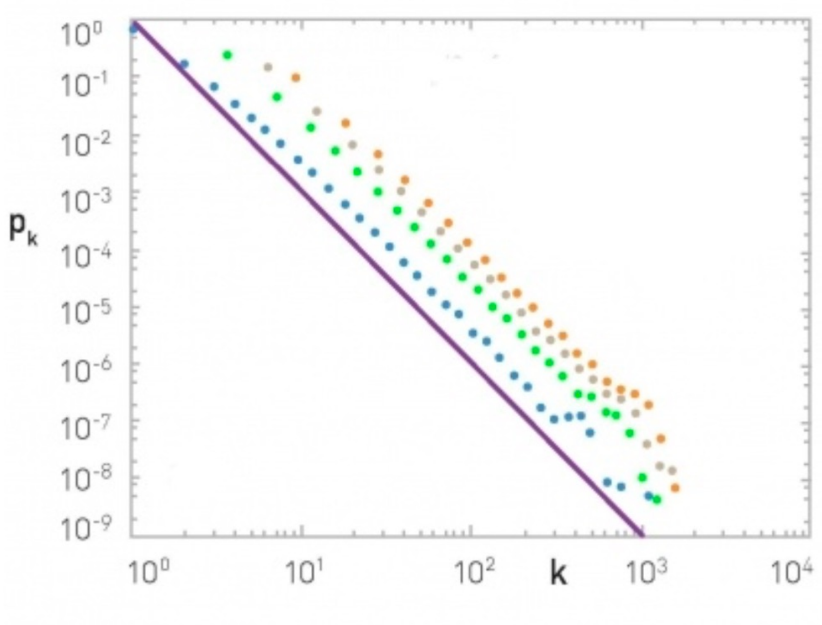
- Note that it does not depend on the time, hence, it describes a stationary network

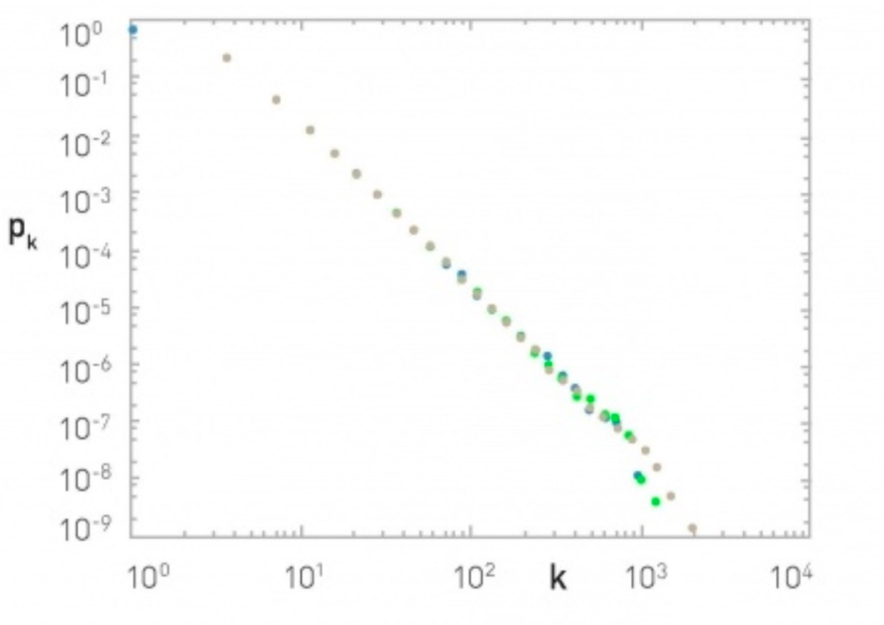
Degree distribution, simulation results

N=100,000
m=3



More simulations





N = 100,000; m_0 = m = 1 (blue), 3 (green)
m=5 (gray), 7 (orange)
Observe γ is independent of m (and m_0)

The slope of the purple line is -3

m_0 = m = 3; N = 50K (blue),
N=100K (green), 200K (gray)

Observe p_k is independent of N

Derivation of the BA degree distribution

Let’s calculate the Cumulative Distribution Function (CDF) of the degree distribution

By definition of CDF, this is equal to:

$$Pr(k_i(t) \leq k) = 1 - Pr(k_i(t) > k)$$

CDF (cont.)

Let's calculate $Pr(k_i(t) > k)$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta}$$

$$k_i(t) > k \Rightarrow m\left(\frac{t}{t_i}\right)^{\beta} > k$$

$$m^{\frac{1}{\beta}}\left(\frac{t}{t_i}\right) > k^{\frac{1}{\beta}}$$

$$\left(\frac{m}{k}\right)^{\frac{1}{\beta}}\left(\frac{t}{t_i}\right) > 1$$

$$\left(\frac{m}{k}\right)^{\frac{1}{\beta}} > \left(\frac{t_i}{t}\right)$$

$$t_i < t\left(\frac{m}{k}\right)^{\frac{1}{\beta}}$$

This means that nodes i with degree larger than k were created at time t_i **before** a certain timestep, which is expected because older nodes have larger degree.

CDF (cont.)

From the previous slide, we have: $Pr(k_i(t) > k) = Pr\left(\left(\frac{m}{k}\right)^{\frac{1}{\beta}} > \frac{t_i}{t}\right)$

Remember there is one node created at each timestep, so by time t there are $N(t) = m_0 + t$ nodes, and for large t, we have $N(t) \approx t$

Now, what is $Pr(x > t_i/t)$ if you pick a node i at random?
It is x, because t_i/t **is distributed uniformly in** $[0,1]$

The cumulative distribution of a uniform distribution is a linear function (think about integration!)

$$Pr(k_i(t) > k) = \left(\frac{m}{k}\right)^{\frac{1}{\beta}}$$

CDF (cont.)

Hence:

$$\begin{aligned} Pr\left(k_i(t) \leq k\right) &= 1 - Pr\left(k_i(t) > k\right) \\ &= 1 - \left(\frac{m}{k}\right)^{\frac{1}{\beta}} \end{aligned}$$

Probability Density Function (PDF)

Now let's take the derivative of the CDF to obtain the PDF

$$p_k = \frac{d}{dk} Pr(k_i \leq k) = \frac{d}{dk}\left(1 - \left(\frac{m}{k}\right)^{1/\beta}\right)$$

$$= -\frac{d}{dk}\left(\left(\frac{m}{k}\right)^{1/\beta}\right) = -m^{1/\beta}\frac{d}{dk}\left(\frac{1}{k^{1/\beta}}\right)$$

$$= \frac{1}{\beta}\frac{m^{1/\beta}}{k^{1/\beta+1}} \qquad (\beta = 1/2)$$

$$= 2\frac{m^2}{k^3}$$

$$p(k) \propto k^{-3}$$

Degree distribution of BA model

- $\beta = 1/2$ is called the dynamical exponent
- $\gamma = \frac{1}{\beta} + 1 = 3$ is the power-law exponent

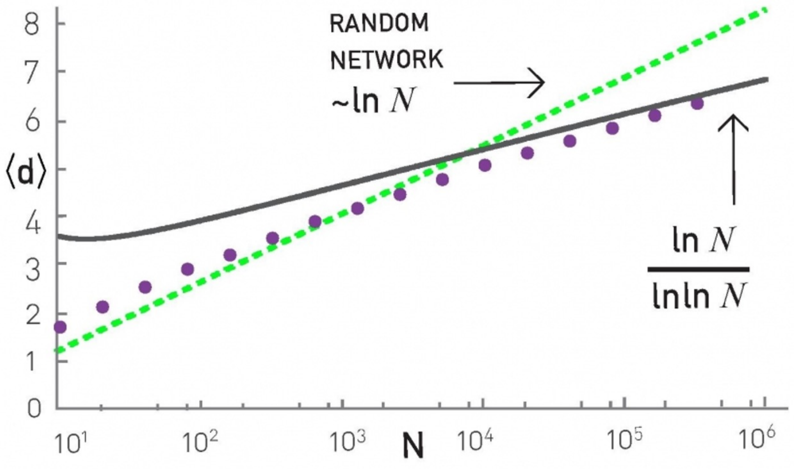
- Note that $p(k) \approx 2m^2/k^3$
 does not depend on t
 hence, it describes a stationary network

Average distance

- Distances grow slower than log N

$$\langle d \rangle \approx \frac{\log N}{\log \log N}$$

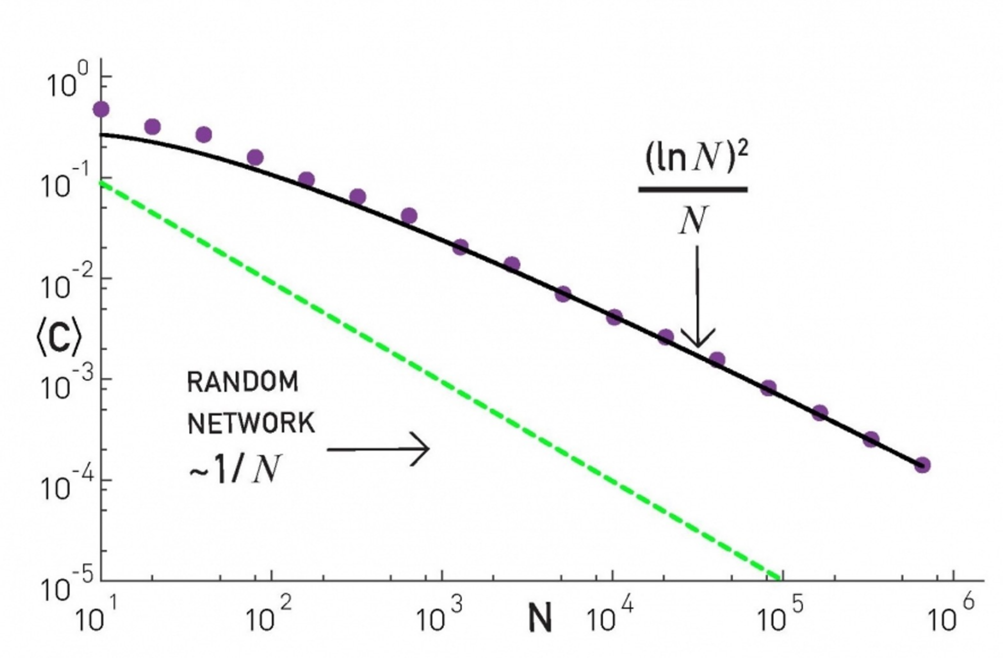
(Why: scale free network with $\gamma = 3$)



Clustering coefficient

- BA networks are locally more clustered than ER networks

$$\langle C \rangle \approx \frac{(\log N)^2}{N}$$



Limitations of the BA model

- Predicts a fixed exponent of -3
- Assumes an undirected network, while many real complex networks are directed
- Does not consider node deletions or edge deletions which are common in practice
- Considers that all nodes are equal except for their arrival times

Summary

Things to remember

- Degree distribution in the BA model
- Distances and clustering coefficient in BA

Practice on your own

- Try to reconstruct the derivations we have done in class; try to understand every step
- Insert a small change in the model and try to recalculate what we have done

Sources

- A. L. Barabási (2016). Network Science – Chapter 05
- R. Srinivasan (2013). Complex Networks – Chapter 12
- D. Easley and J. Kleinberg (2010): Networks, Crowds, and Markets – Chapter 18
- Data-Driven Social Analytics course by Vicenç Gómez and Andreas Kaltenbrunner