# Betweenness

## Social Networks Analysis and Graph Algorithms

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# Sources

- D. Easly and J. Kleinberg (2010). Networks, Crowds, and Markets – Section 3.6B
- A. L. Barabási (2016). Network Science – Section 9.3
- P. Boldi and S. Vigna (2014). Axioms for Centrality in *Internet Mathematics*
- Esposito and Pesce: Survey of Centrality 2015.
- URLs cited in the footer of slides

# Types of centrality measure

- **Non-spectral**
  - Degree
  - Closeness and harmonic closeness
  - Betweenness
- Spectral
  - HITS
  - PageRank

# Betweenness

# Definitions

The **betweenness of a node** is the number of shortest paths that cross that node

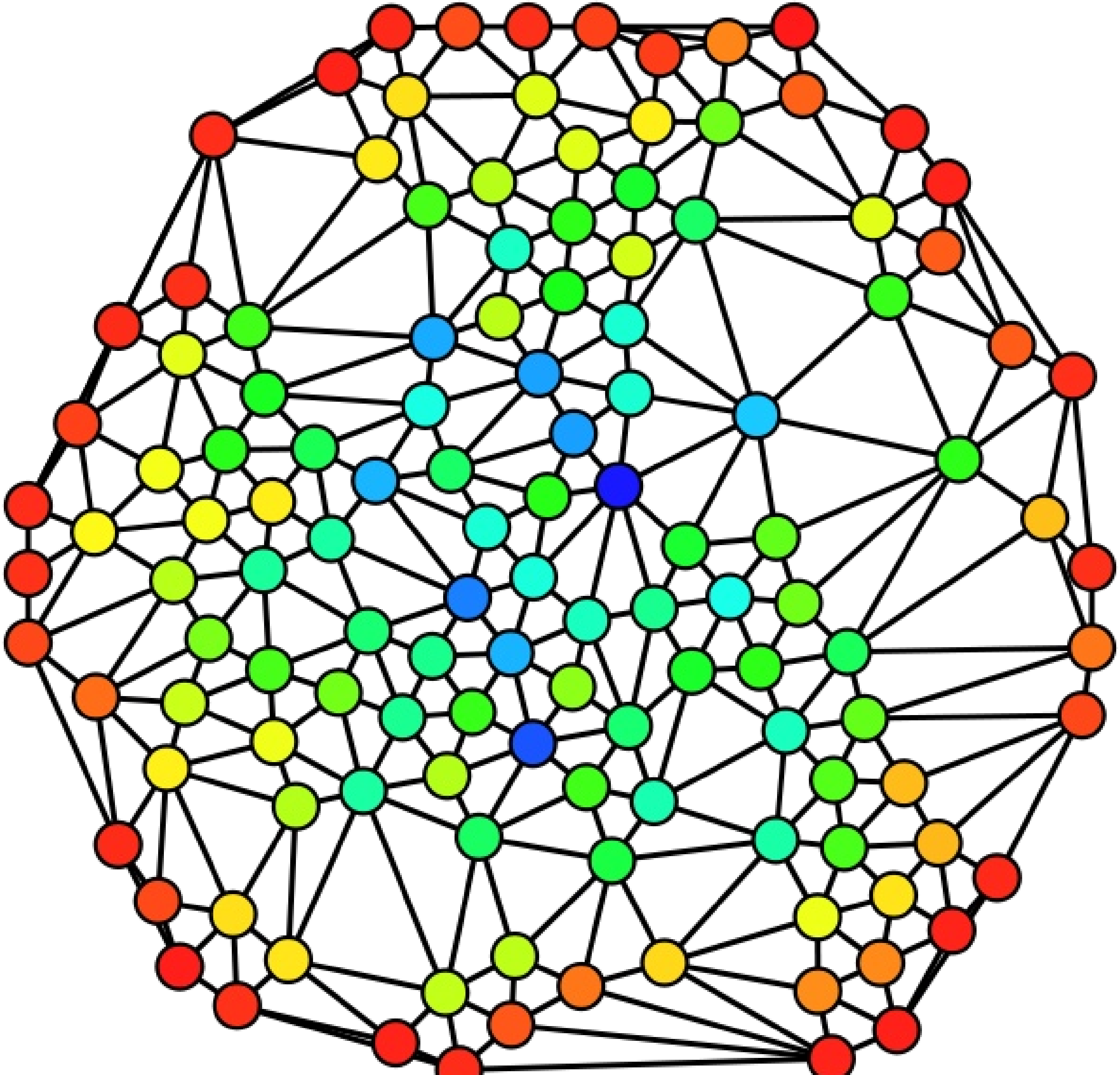
The **betweenness of an edge** is the number of shortest paths that cross that edge

# Node Betweenness

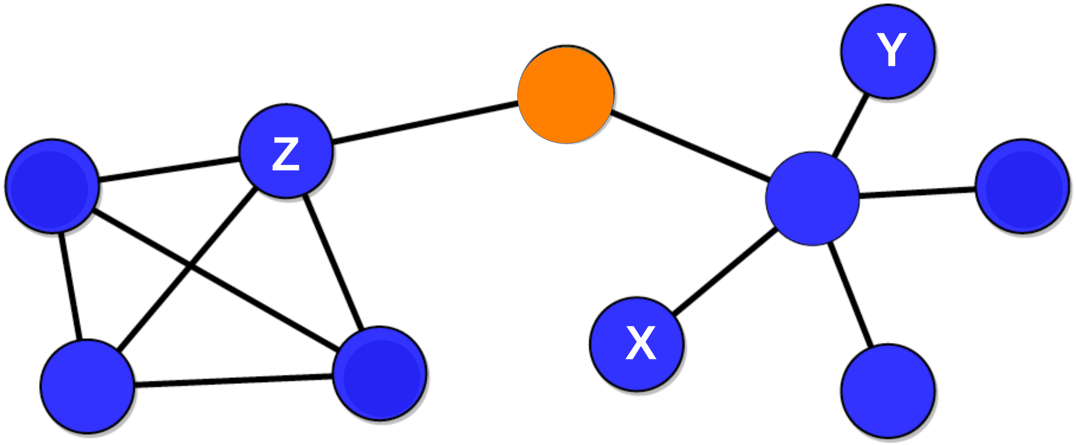
Graph with nodes colored according to node betweenness

red=low, blue=high

[Wikipedia: Betweenness centrality]



# Example 1



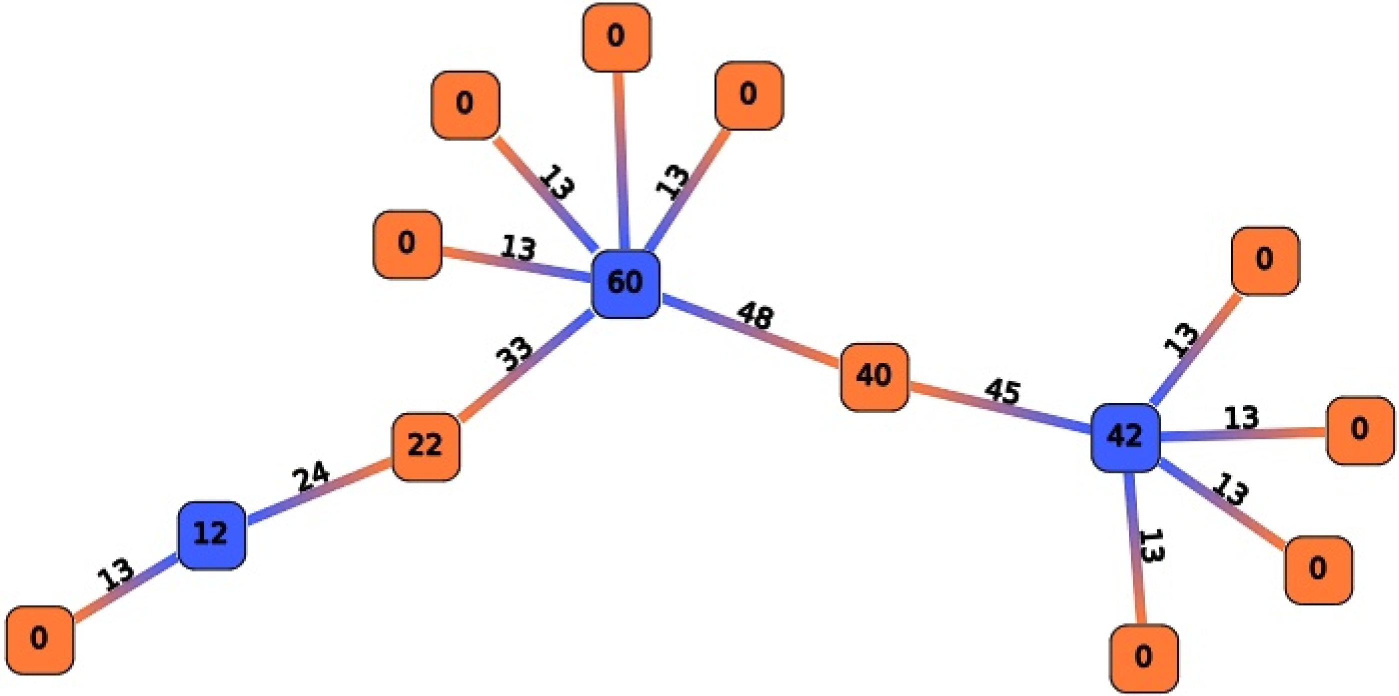

There are 20 shortest paths that cross through the orange node. Why?

The shortest path between nodes X and Y does not cross the orange node, but the shortest path between nodes X and Z does cross the orange node.

Source: Menczer, Fortunato, Davis: A First Course on Networks Science. Cambridge, 2020.

# Example 2

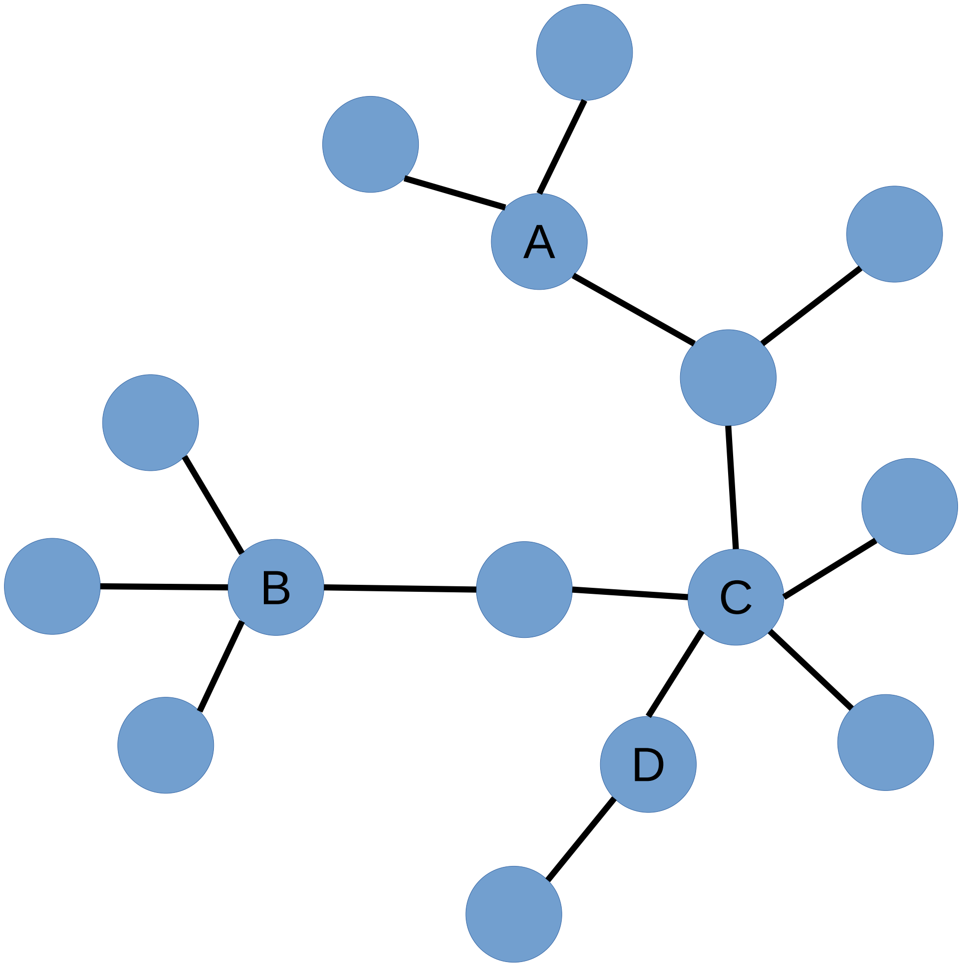
Here, nodes and edges are labeled with their betweenness.




http://rnav.labri.fr/rNAV2.0_public_version/online_doc/_images/ComputeBetweennessCentrality.png

# Exercise

Compute the node betweenness of the nodes marked with letters.

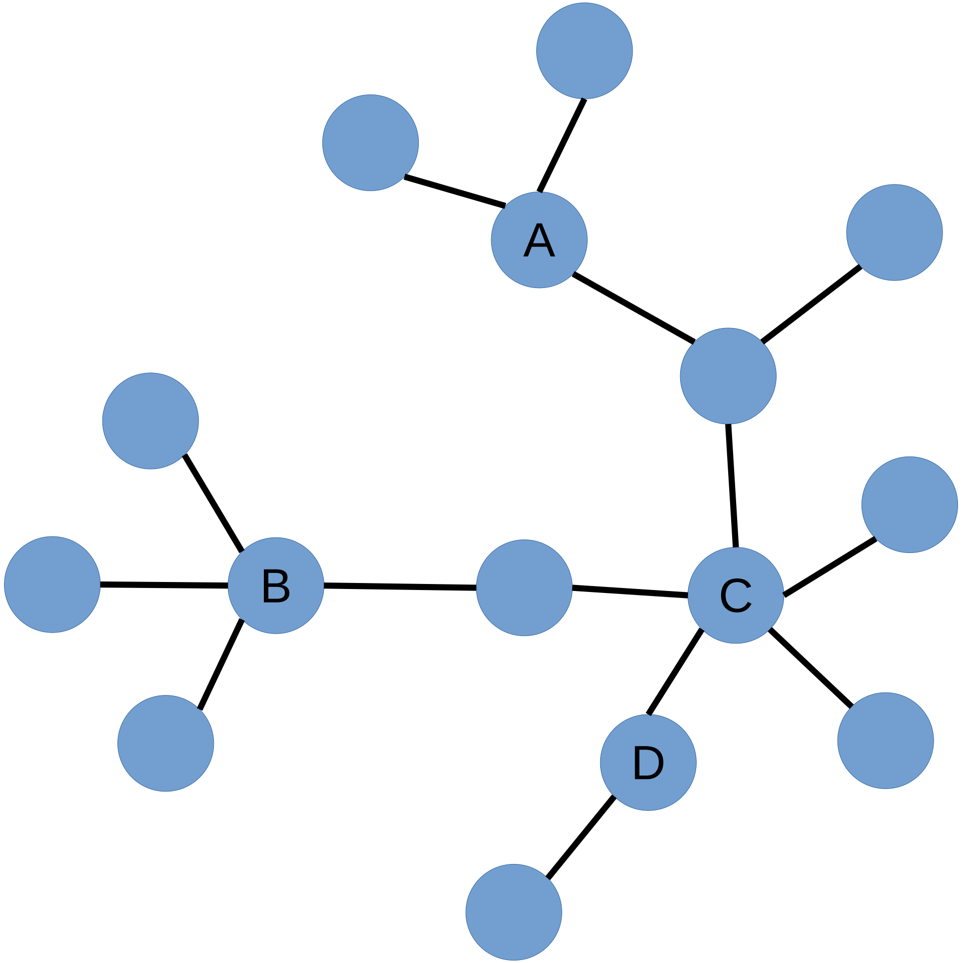



Pin board: https://upfbarcelona.padlet.org/chato/asfs154waxnnkhgo

# Exercise (cont.)

What is a good algorithm to compute node betweenness of all nodes?

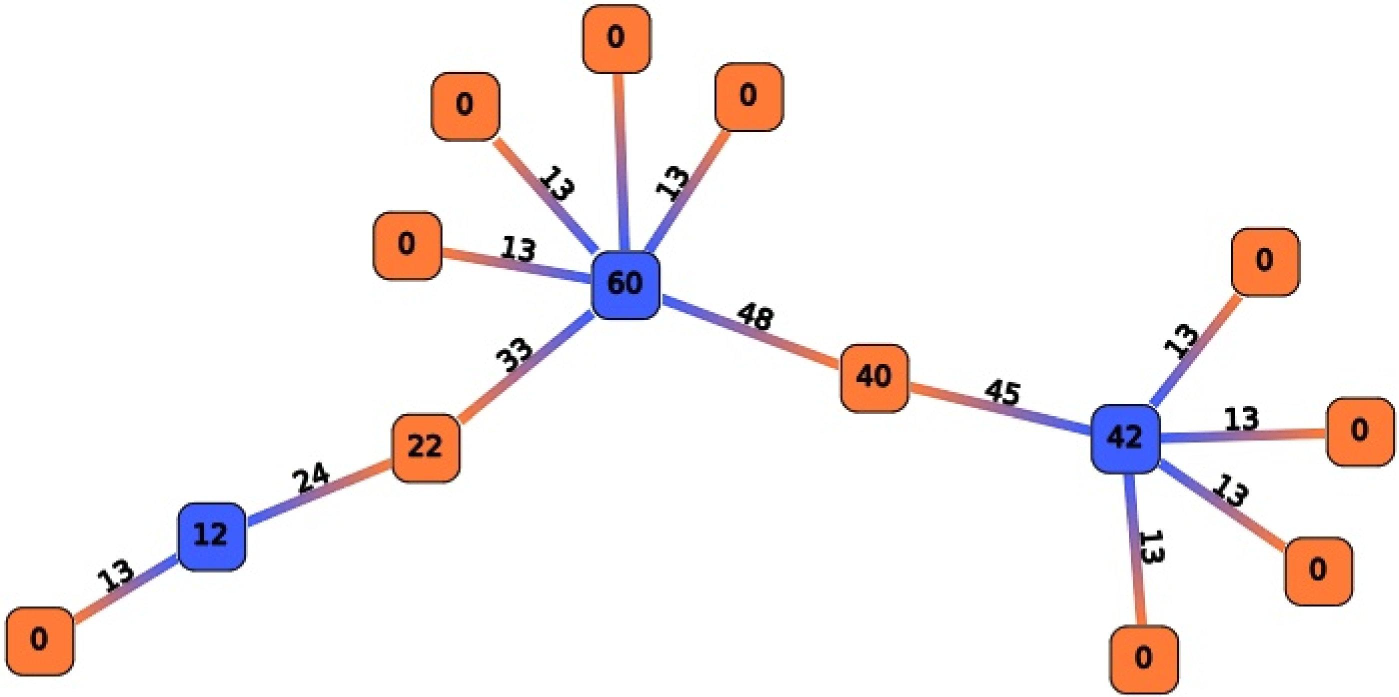
What limitations does your algorithm have?



# Edge Betweenness

# Edge Betweenness

An **edge** has high betweenness if it is part of many shortest-paths.




http://rnav.labri.fr/rNAV2.0_public_version/online_doc/_images/ComputeBetweennessCentrality.png

# Approximate method [sampling]

- Label all edges e with $b(e) = 0$
- Repeat K times:
  - Pick a random pair of nodes (u,v)
  - Compute shortest path between u and v
  - $b(e) \leftarrow b(e) + 1$ for all edges e along the path
- b(e) is a lower bound for betweenness (e)
- Useful if we only care about finding the edge with the highest betweenness, or finding the top-k edges with the highest betweenness $\rightarrow$ an early stopping criterion is possible

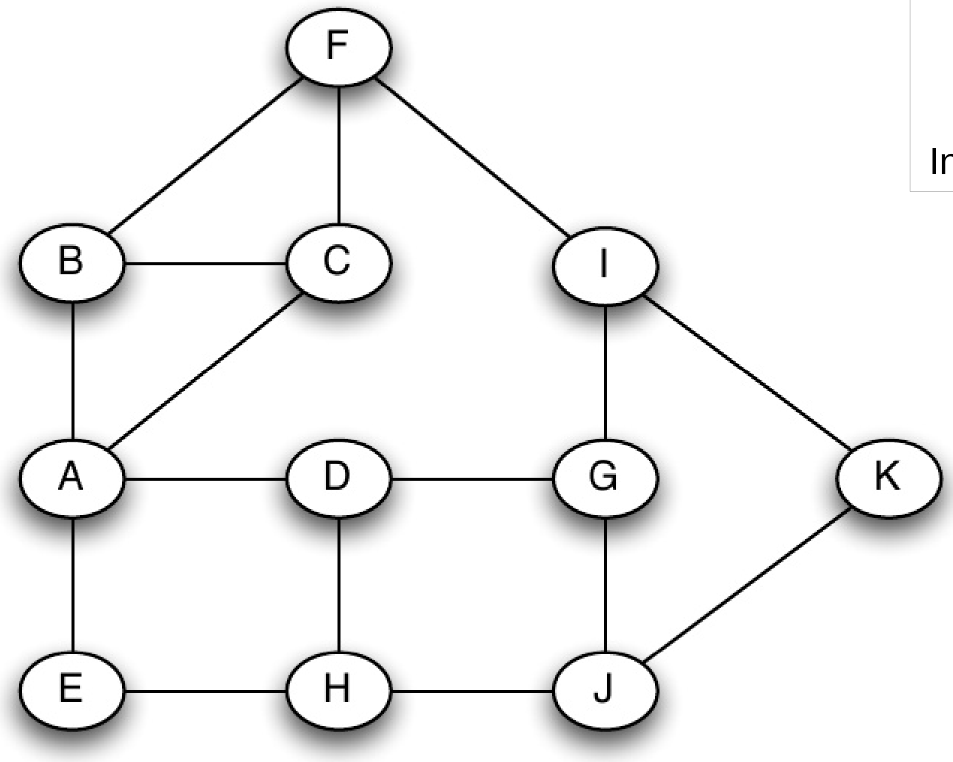
Riondato, M., & Kornaropoulos, E. M. (2014, February). Fast approximation of betweenness centrality through sampling. In Proceedings of the 7th ACM international conference on Web search and data mining (pp. 413-422).

# Exact algorithm [Brandes, Newman]

- For every node $u$ in $V$
  - Layer the graph performing a BFS from $u$
  - For every node $v$ in $V$, $v \neq u$, sorted by layer
    - Assign to $v$ a number $s(v)$ indicating how many shortest paths from u arrive to v
  - For every node $v$ in $V$, $v \neq u$, sorted by reverse layer
    - Score to distribute $= 1 +$ score from children
    - Add score to parent edges in proportion to $s(v)$
- In the end divide all edge scores by two

Ulrich Brandes. A faster algorithm for betweenness centrality. Journal of Mathematical Sociology, 25:163–177, 2001.
Mark E. J. Newman. Scientific collaboration networks: II. Shortest paths, weighted networks, and centrality. Physical Review E, 64:016132, 2001.

# Example

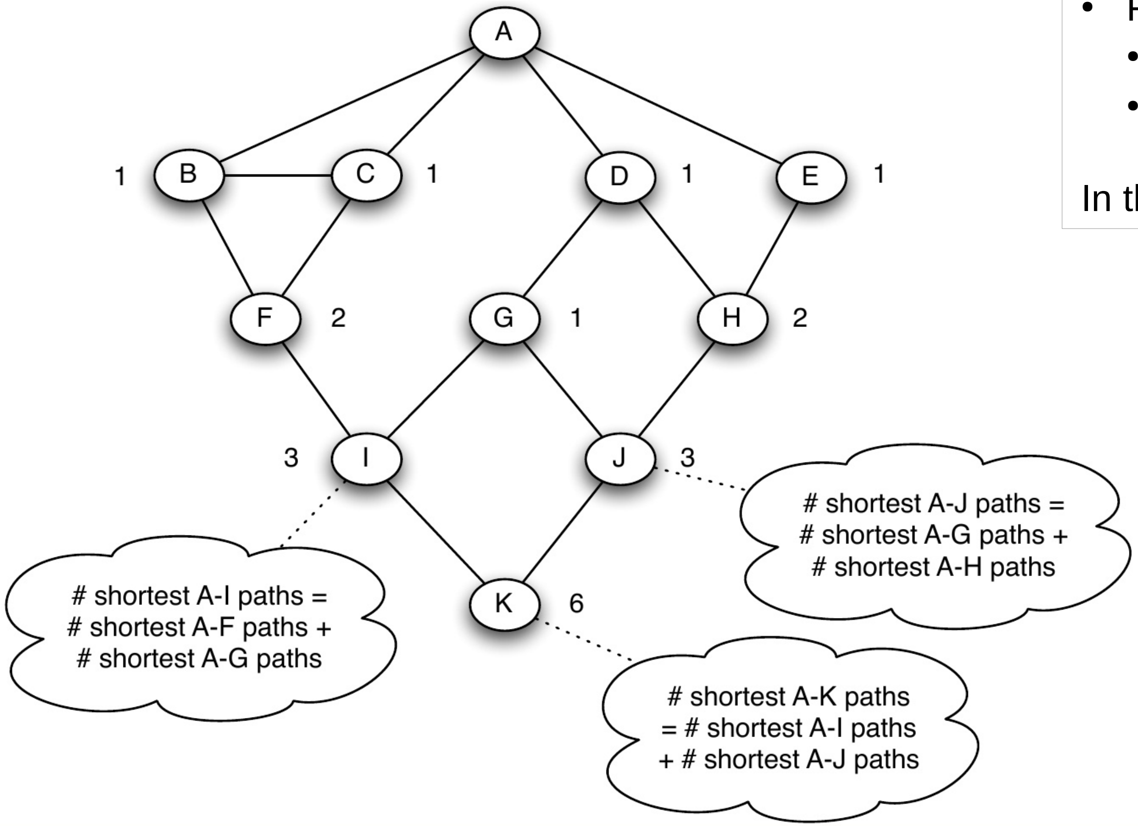



For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
  - Assign to v a number s(v) indicating how many shortest paths from u arrive to v
- For every node v in V, v≠u, sorted by reverse layer
  - Score to distribute = 1 + score from children
  - Add score to distribute to parent edges in proportion to s(v)

In the end divide all edge scores by two

# Example




For every node u in V
- Layer the graph performing a BFS from u
- **For every node v in V, v≠u, sorted by layer**
  - **Assign to v a number s(v) indicating how many shortest paths from u arrive to v**
- For every node v in V, v≠u, sorted by reverse layer
  - Score to distribute = 1 + score from children
  - Add score to distribute to parent edges in proportion to s(v)

In the end divide all edge scores by two

All nodes in layer 1 get s(v)=1

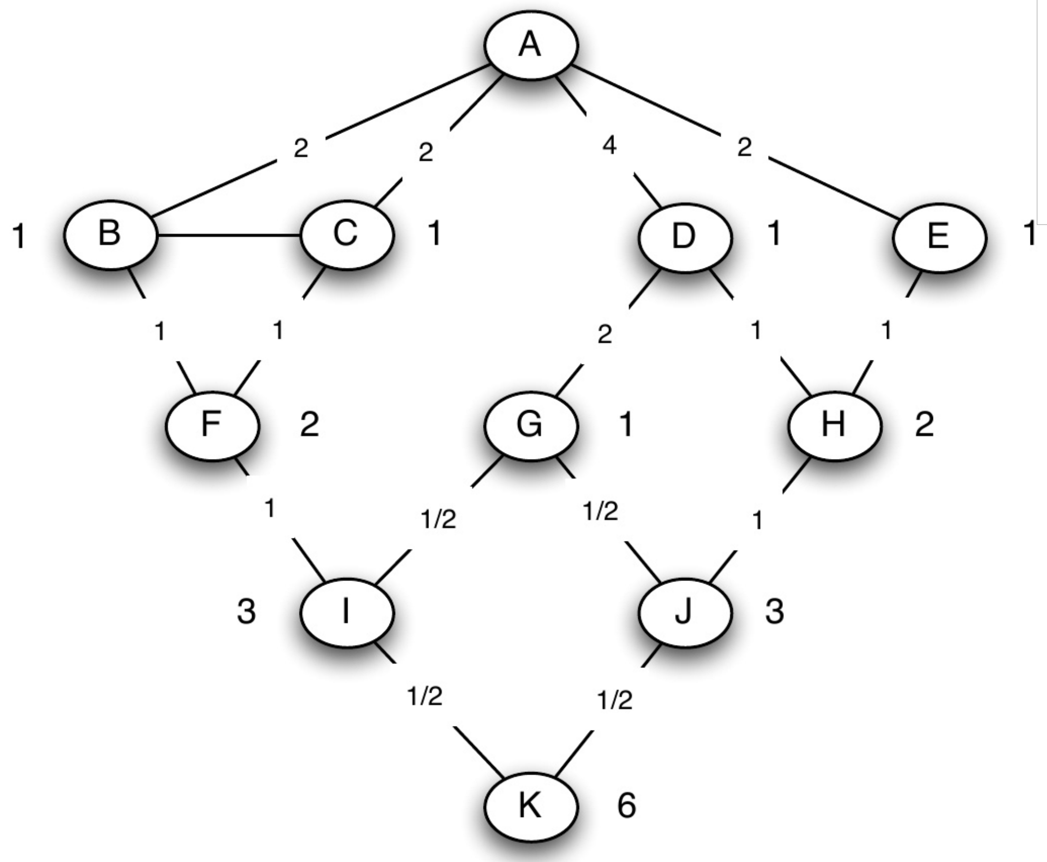
Remaining nodes: simply add s(.) of their parents

# Example

For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
  - Assign to v a number s(v) indicating how many shortest paths from u arrive to v
- **For every node v in V, v≠u, sorted by rev. layer**
  - **Score to distribute = 1 + score from children**
  - **Add score to distribute to parent edges in proportion to s(v)**

In the end divide all edge scores by two




Nodes without children distribute a score of 1

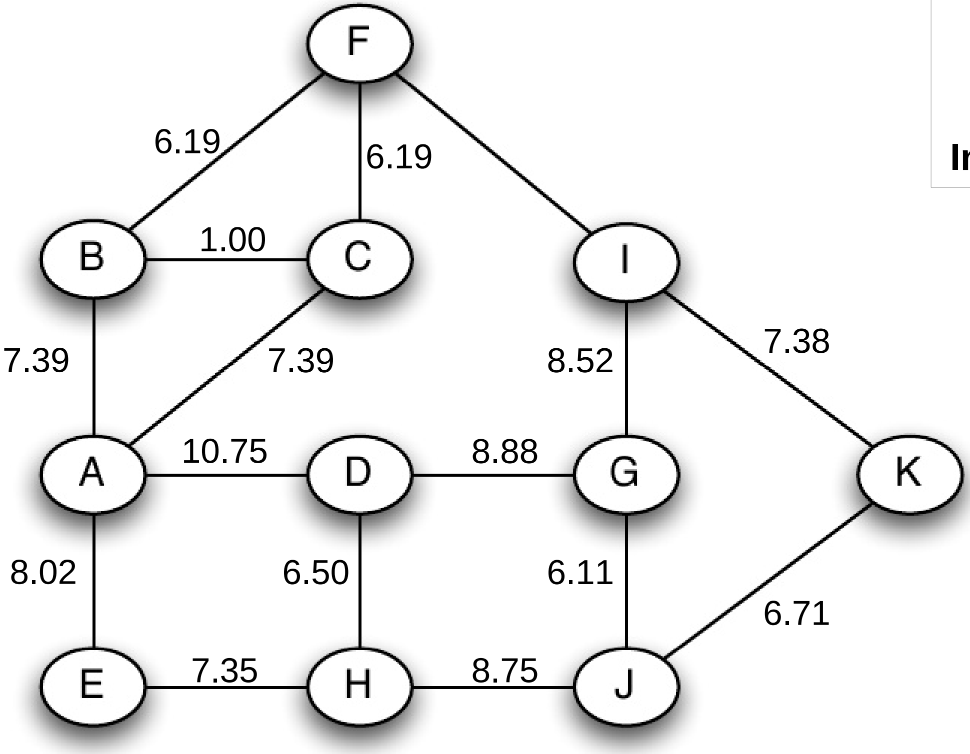
Other nodes distribute 1 + whatever they receive from their children

# Result

For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
  - Assign to v a number s(v) indicating how many shortest paths from u arrive to v
- For every node v in V, v≠u, sorted by reverse layer
  - Score to distribute = 1 + score from children
  - Add score to distribute to parent edges in proportion to s(v)

**In the end divide all edge scores by two**

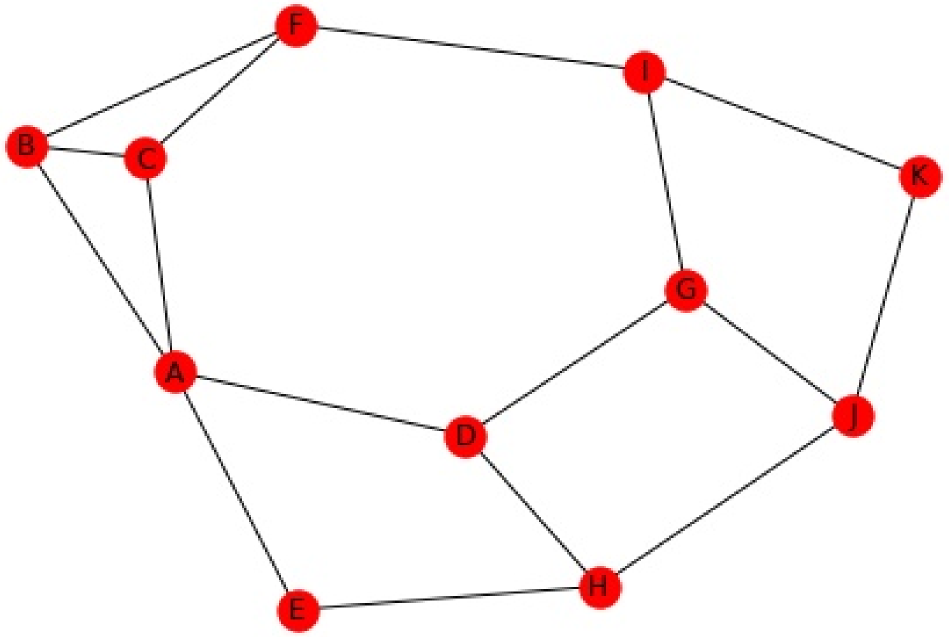



Computed using NetworkX (edge betweenness)

# NetworkX code

```
import networkx as nx
g = nx.Graph()
g.add_edge("A", "B")
g.add_edge("A", "C")
g.add_edge("A", "D")
g.add_edge("A", "E")
g.add_edge("B", "C")
g.add_edge("B", "F")
g.add_edge("C", "F")
g.add_edge("D", "G")
g.add_edge("D", "H")
g.add_edge("E", "H")
g.add_edge("F", "I")
g.add_edge("G", "I")
g.add_edge("G", "J")
g.add_edge("H", "J")
g.add_edge("I", "K")
g.add_edge("J", "K")
nx.edge_betweenness(g, normalized=False)
```

```
nx.draw_spring(g, with_labels=True)
```



# Exercise

*Try to compute **edge betweenness** by inspection first*

*Then use the Brandes-Newman algorithm; you should get the same results*

For every node u in V
- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
  - Assign to v a number s(v) indicating how many shortest paths from u arrive to v
- For every node v in V, v≠u, sorted by reverse layer
  - Score to distribute = 1 + score from children
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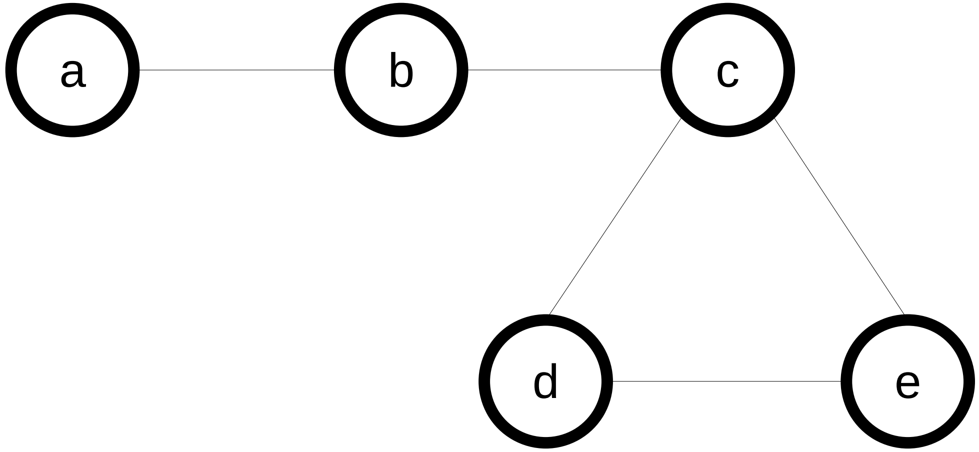
In the end divide all edge scores by two



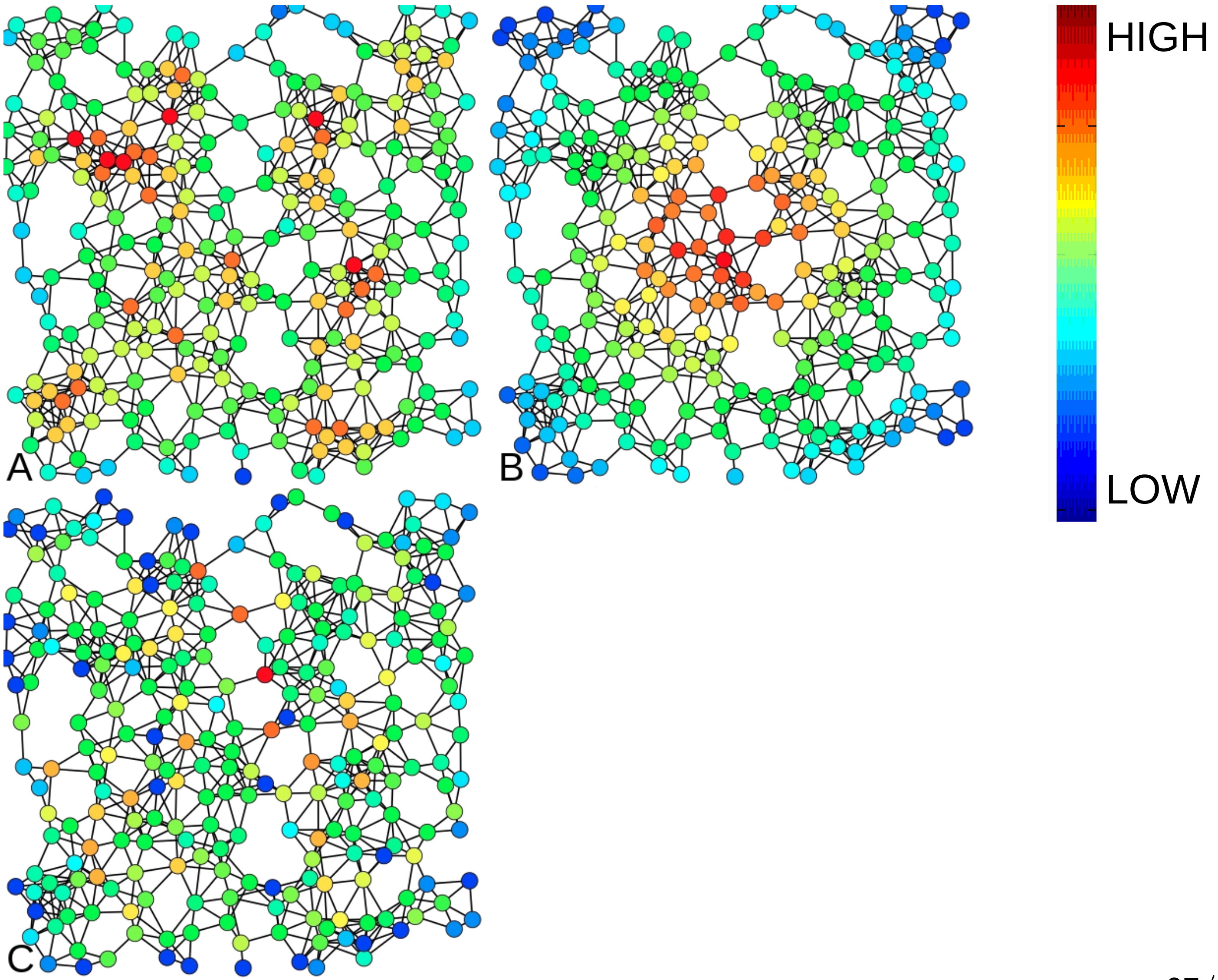
# Fractional values?

- In a graph with cycles, you may get **fractional values** of the edge betweenness for an edge
- Conceptually, this is because in a graph with cycles there might be s$>$1 shortest paths between two nodes, each of them counts $1/s$

A: Degree

B: Closeness

C: Betweenness



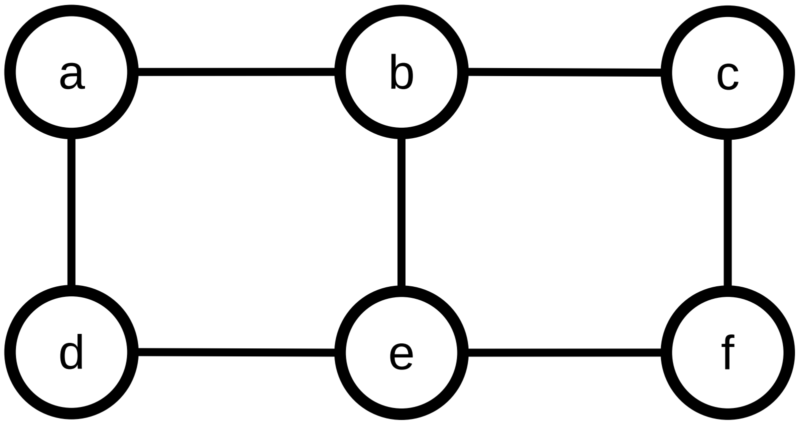
# Summary

# Things to remember

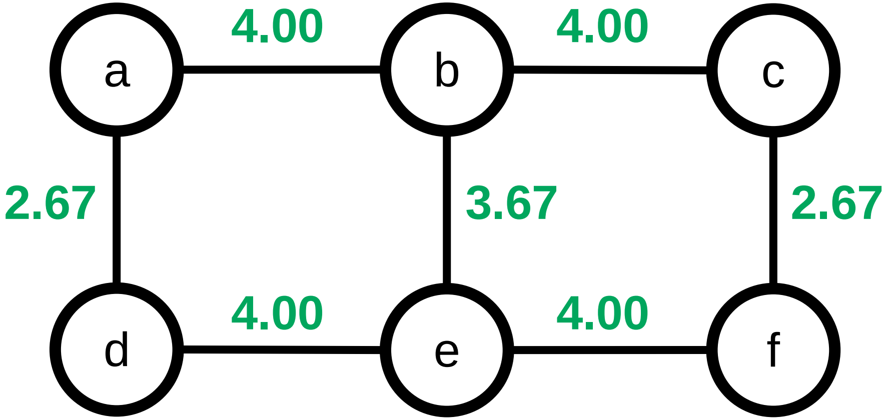
- Closeness and harmonic closeness
- Node and edge betweenness
- Practice running the Brandes-Newman algorithm on small graphs
- Write code to execute the Brandes-Newman algorithm

# Practice on your own

- Compute edge betweenness on this graph



# Practice on your own (cont.)




If you don't get this result, check:

https://www.youtube.com/watch?v=uYjWbp8VC7c

# Two constructive problems

1. Sketch a graph of $N$ nodes in which a node, which you should mark with an asterisk (*), should have betweenness approximately equal to $N$ and closeness approximately $1/N$ for large $N$. Explain briefly.
2. Sketch a graph of $N$ nodes in which a node, which you should mark with an asterisk (*), should have betweenness approximately equal to $N$ and closeness approximately $2/N^2$ for large $N$. Explain briefly.

*Do not use a concrete $N$. Use a general $N$, for instance by using the ellipsis ($\ldots$) to denote multiple nodes.*