Graph Theory Basics

Social Networks Analysis and Graph Algorithms

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Contents

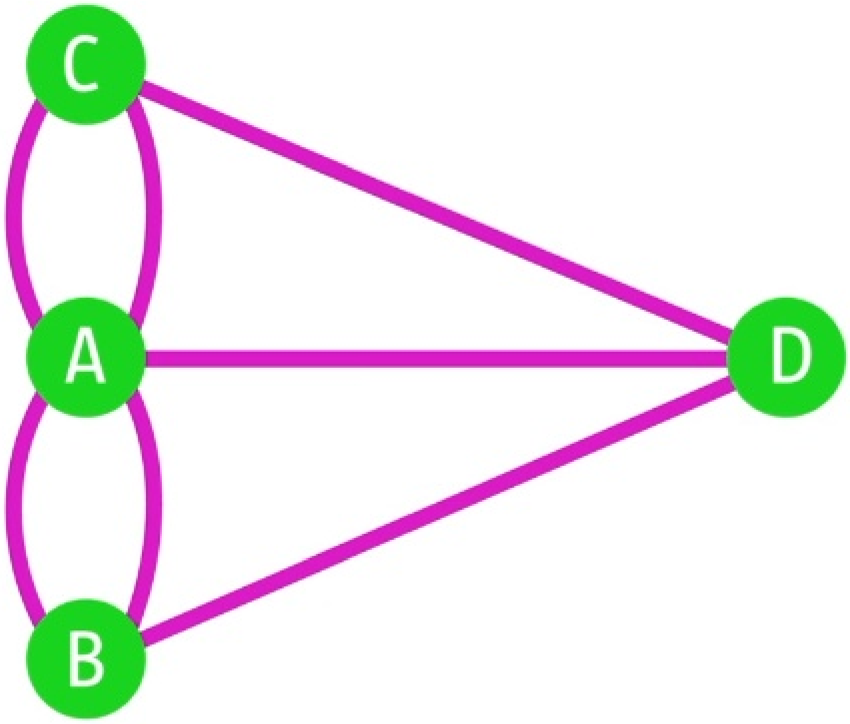
- Notation for graphs
- Degree distributions
- Adjacency matrices

Sources

- A. L. Barabási (2016). Network Science – Chapter 02
- URLs cited in the footer of specific slides

Notation for a graph

- G = (V,E)
  - V: nodes or vertices
  - E: links or edges
- |V| = N size of graph
- |E| = L number of links



Subgraph

- Given $G = (V,E)$
- A **subgraph** induced by a nodeset S is the graph $G' = (S,F)$ defined by:

  nodes in S

  edges in $F = \{ (u,v) \in E \text{ s.t. } u \in S \text{ and } v \in S \}$

Typical notation variations

- You may find that G is denoted by (N, A), this is typical of directed graphs, means "*nodes, arcs*"
- You may find that
  - $|V|$ is denoted by n or N
  - $|E|$ is denoted by m, M, or L

Example graphs we will use

Network	**|V|**	**|E|**
Zachary's Karate Club (karate.gml)	34	78
Game of Thrones (got-relationships.csv)	84	216
US companies ownership	1351	6721
Marvel comics (hero-network.csv)	6K	167K

https://github.com/chatox/networks-science-course/tree/master/practicum/data

Directed vs undirected graphs

- In an undirected graph
  - E is a symmetric relation

    $$(u, v) \in E \Rightarrow (v, u) \in E$$

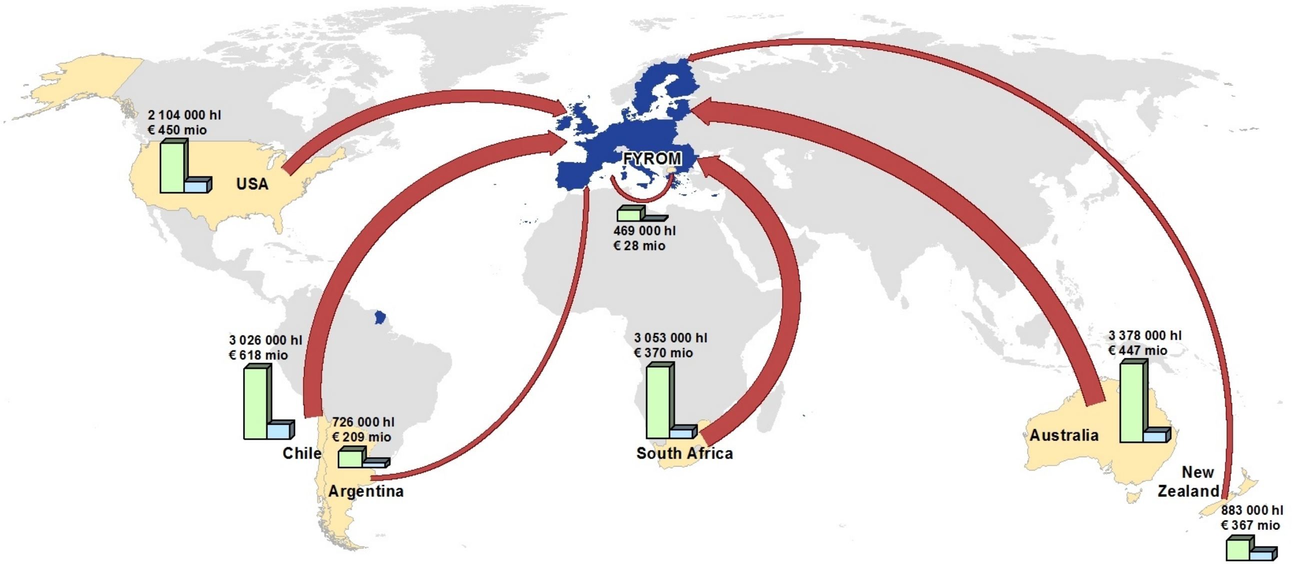
- In a directed graph, also known as “digraph”
  - E is not a symmetric relation

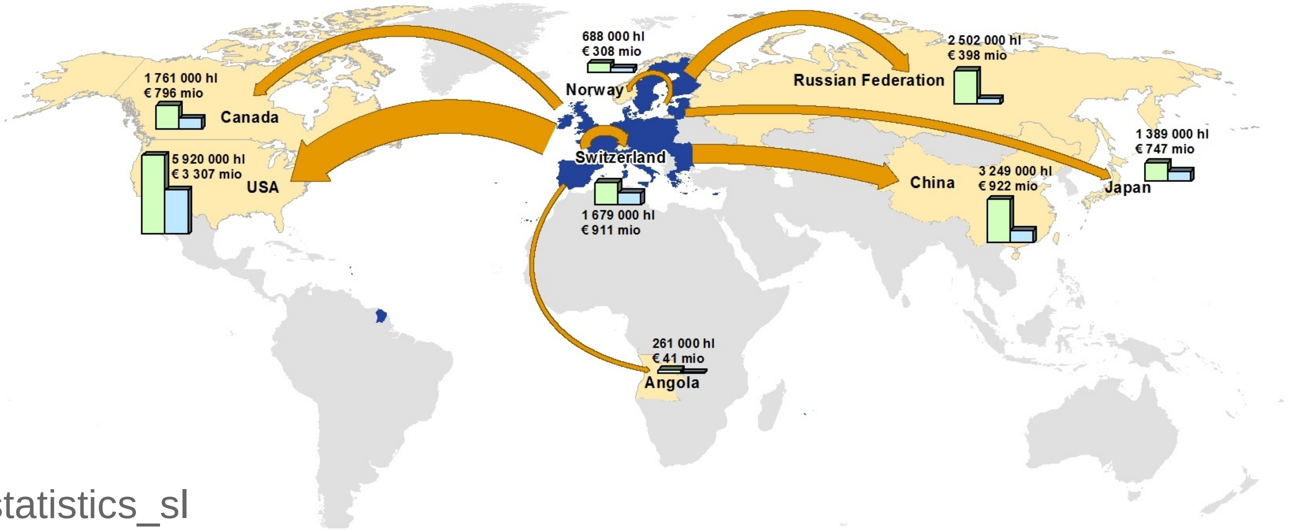
    $$(u, v) \in E \nRightarrow (v, u) \in E$$

Weighted vs unweighted graphs

- In a weighted graph edges have **weights** denoting the strength or importance of a connection
- When drawing, typically larger weights are drawn with ticker lines

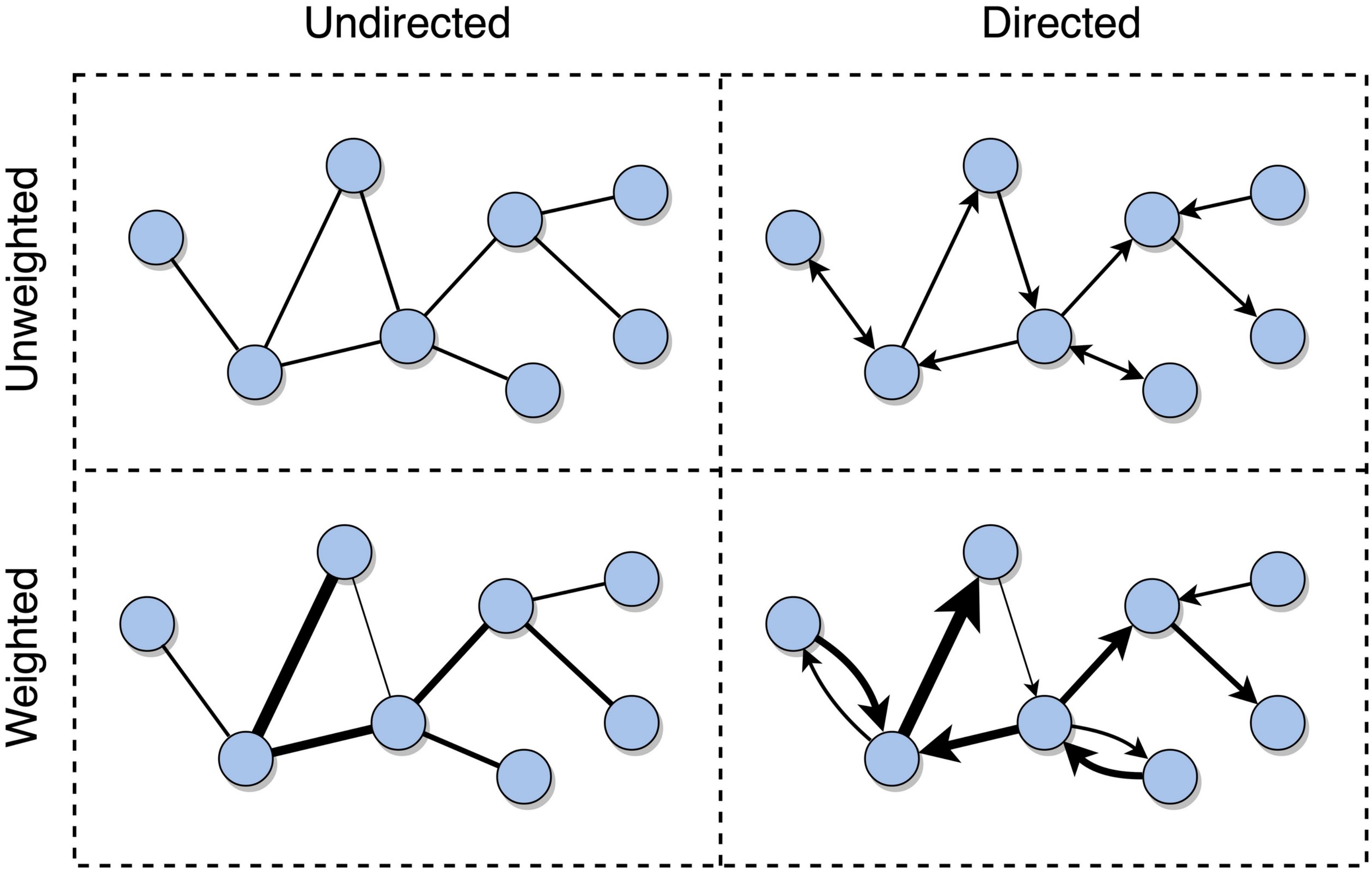
Example: weighted networks





EU imports (top) and exports (bottom) of wine

https://ec.europa.eu/agriculture/wine/statistics_sl



Source: Menczer, Fortunato, David: A First Course on Networks Science. Cambridge, 2020.

Degree

Degree

- Node i has degree k_i
  - This is the number of links incident on this node
  - The total number of links L is given by

$$L = \frac{1}{2}\sum_{i=1}^{N} k_i$$

- Average degree

$$\langle k \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{2L}{N}$$

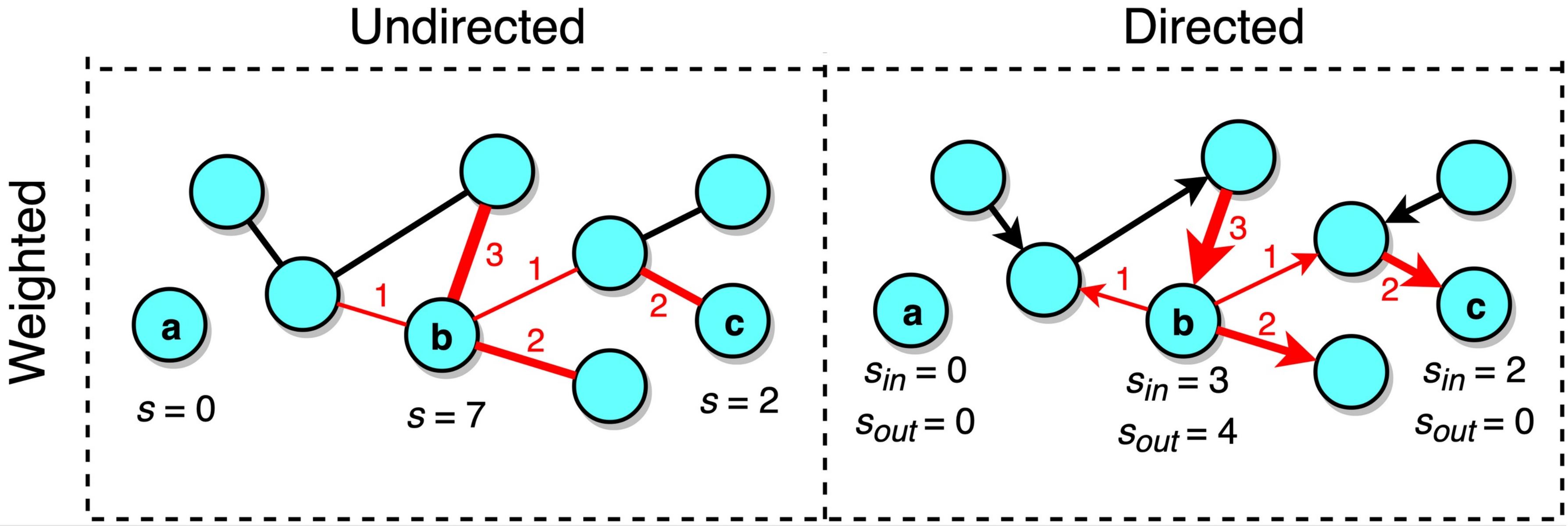
In directed graphs

- We distinguish in-degree from out-degree
  - Incoming and outgoing links, respectively
- Degree is the sum of both $k_i = k_i^{\text{in}} + k_i^{\text{out}}$
- Counting total number of links:

$$L = \sum_{i=1}^{N} k_i^{\text{in}} = \sum_{i=1}^{N} k_i^{\text{out}}$$

In weighted graphs

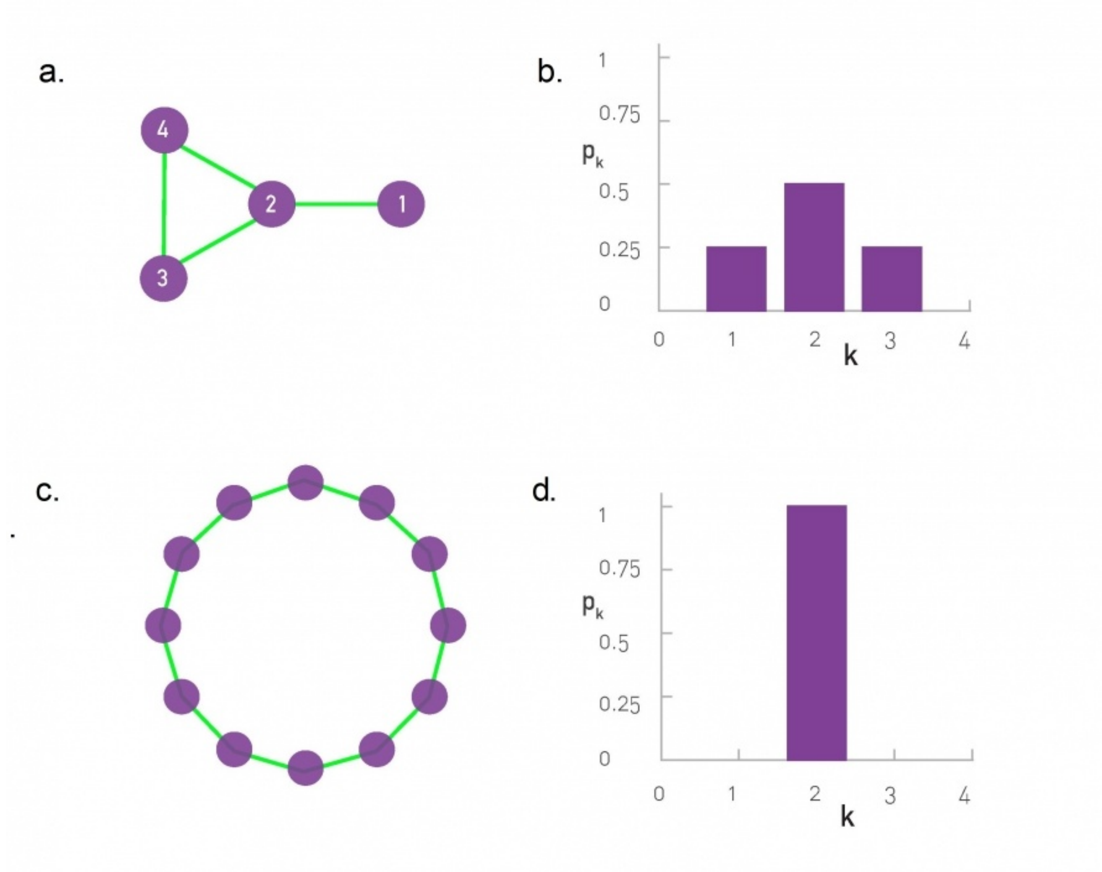
We speak of “weighted degree” or “strength”



Degree distribution

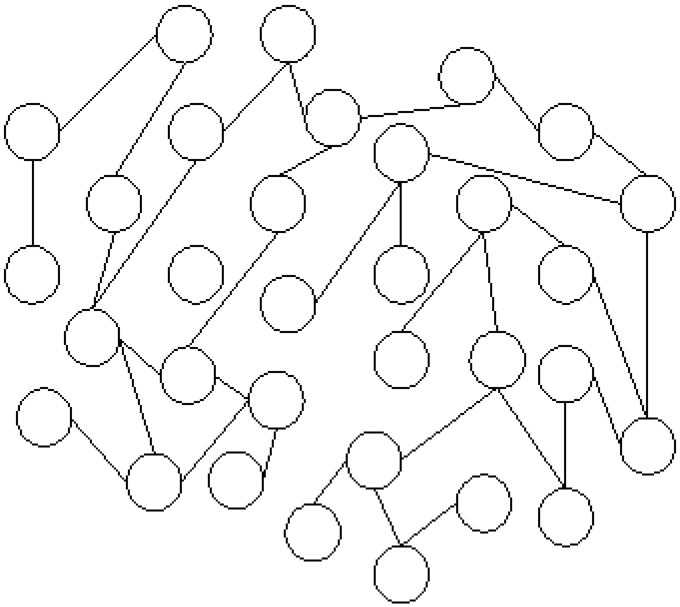
- If there are N_k nodes with degree k
- The degree distribution is given by $p_k = \dfrac{N_k}{N}$
- The average degree is then $\langle k \rangle = \sum_{k=0}^{\infty} k p_k$

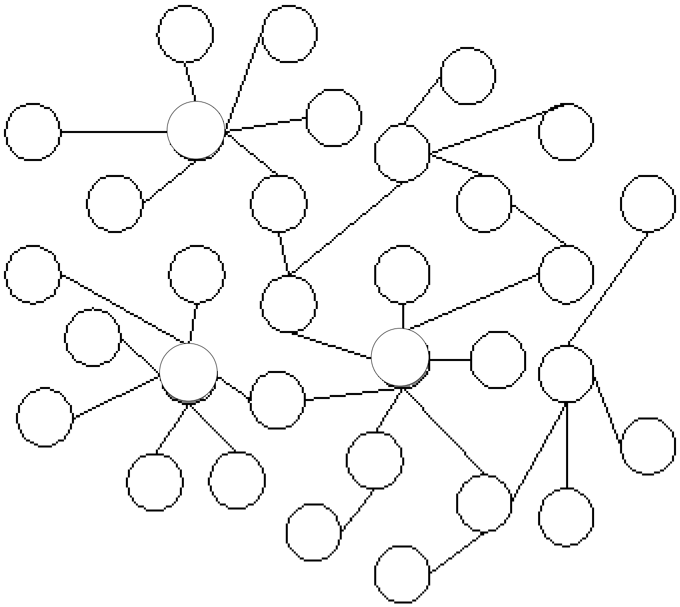
Degree distribution; two toy graphs



Exercise

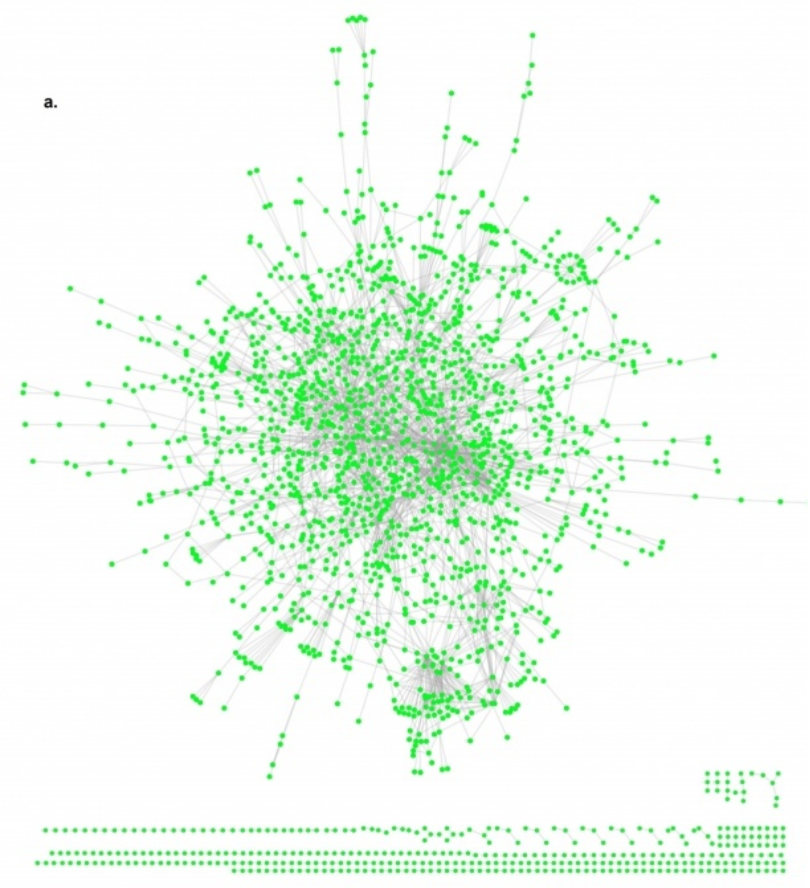
Draw the degree distribution of these graphs

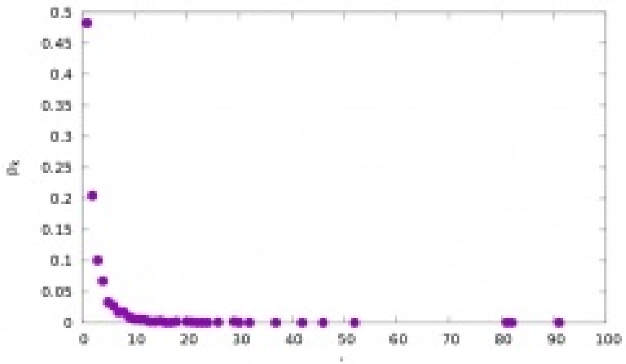




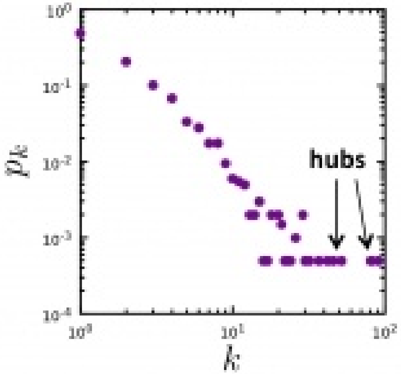
Spreadsheet links: https://upfbarcelona.padlet.org/chato/shyq9m6f2g2dh1bw

Degree distribution; real graph





Linear scale



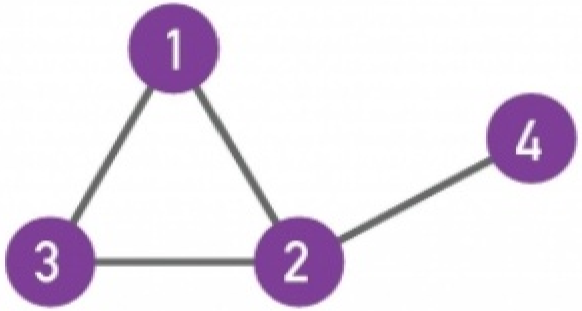
Log-log scale

Adjacency matrix

What is an adjacency matrix

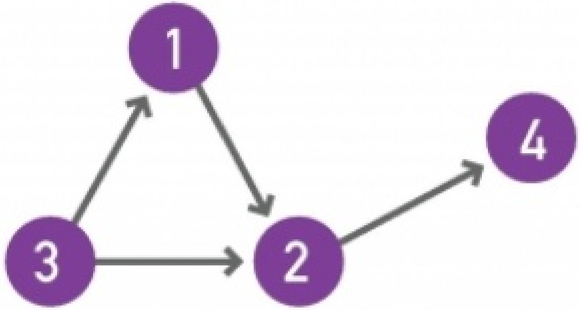
- A is the adjacency matrix of $G = (V, E)$ iff:
  - A has $|V|$ rows and $|V|$ columns
  - $A_{ij} = 1$ if $(i,j) \in E$
  - $A_{ij} = 0$ if $(i,j) \notin E$
- **A_{ij} always means row i, column j**
  - Sometimes Barabási's book has this wrong

Examples



Undirected graph

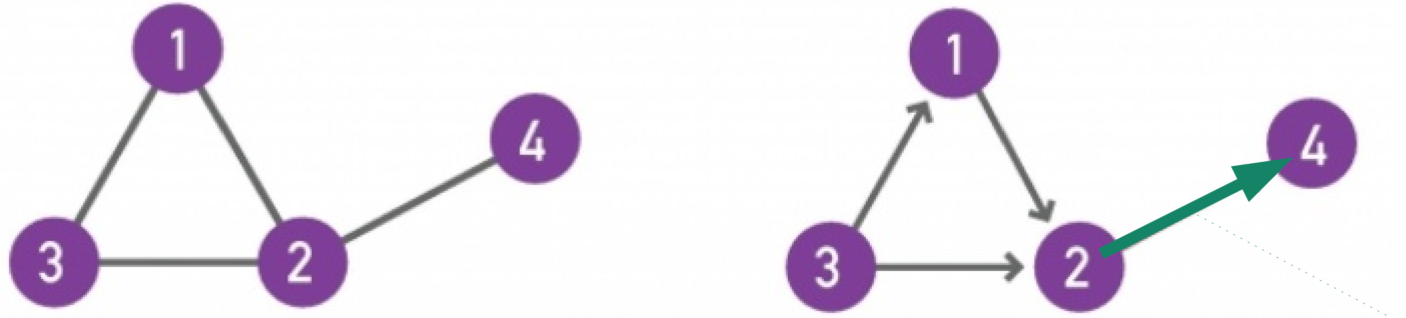
$$A_{ij} = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



Directed graph

$$A_{ij} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

A_{ij} always means row i, column j



Undirected graph

$$A_{ij} = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

Directed graph

Row 2
Column 4

$$A_{ij} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Properties of adjacency matrices

- G is undirected $\Leftrightarrow$ A is symmetric
- G has a self-loop

  $\Leftrightarrow$ A has a non-zero element in the diagonal
- G is complete $\Leftrightarrow$ $A_{ij} \neq 0$ (except if i=j)

Quick Exercise

- In terms of A, what is the expression for:

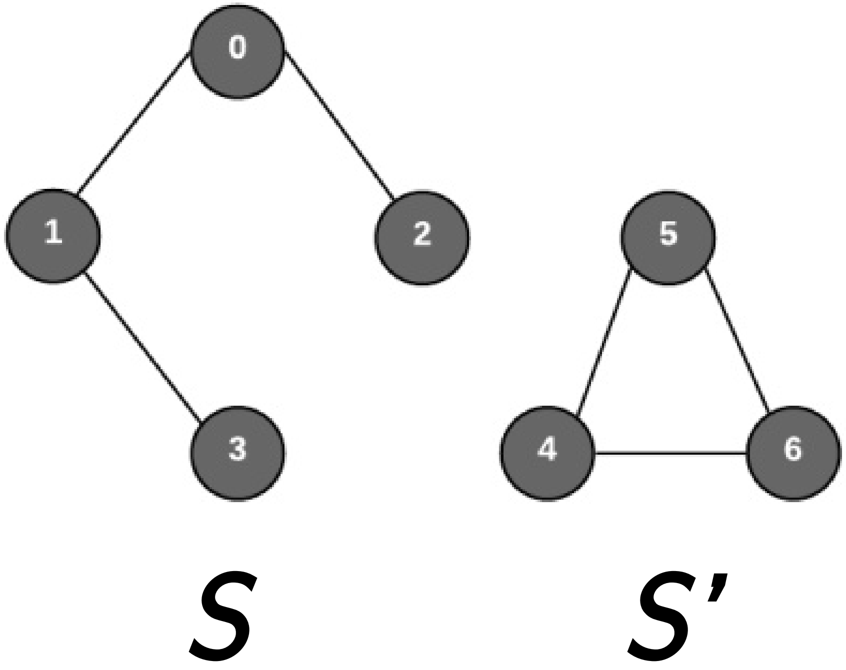
$$k_i^{\text{in}} =$$

$$k_i^{\text{out}} =$$

If a graph is disconnected

Disconnected graphs have adjacency matrices with **block structure**

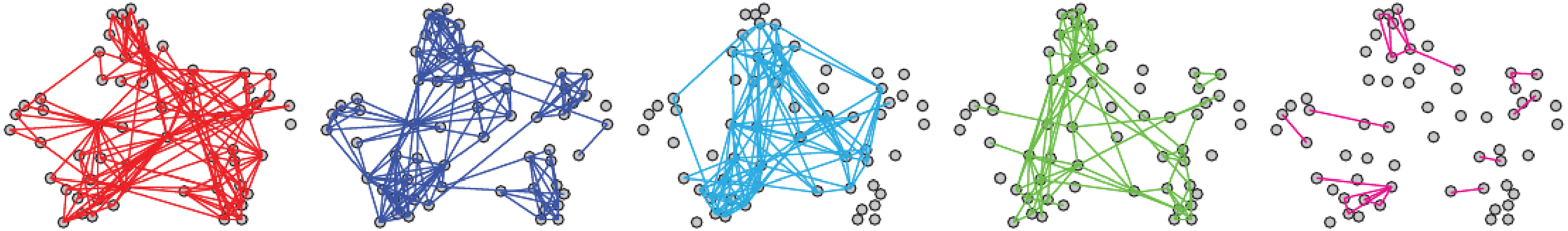
$$A = \begin{bmatrix} S & 0 \\ 0 & S' \end{bmatrix}$$



More concepts

Some graphs are multi-layer

- Multi-layer graphs have different edges **over the same nodes**
- Example graph of 61 employees of a university department, with links indicating: coworking, having lunch together, being connected on Facebook, being actual friends, or being co-authors



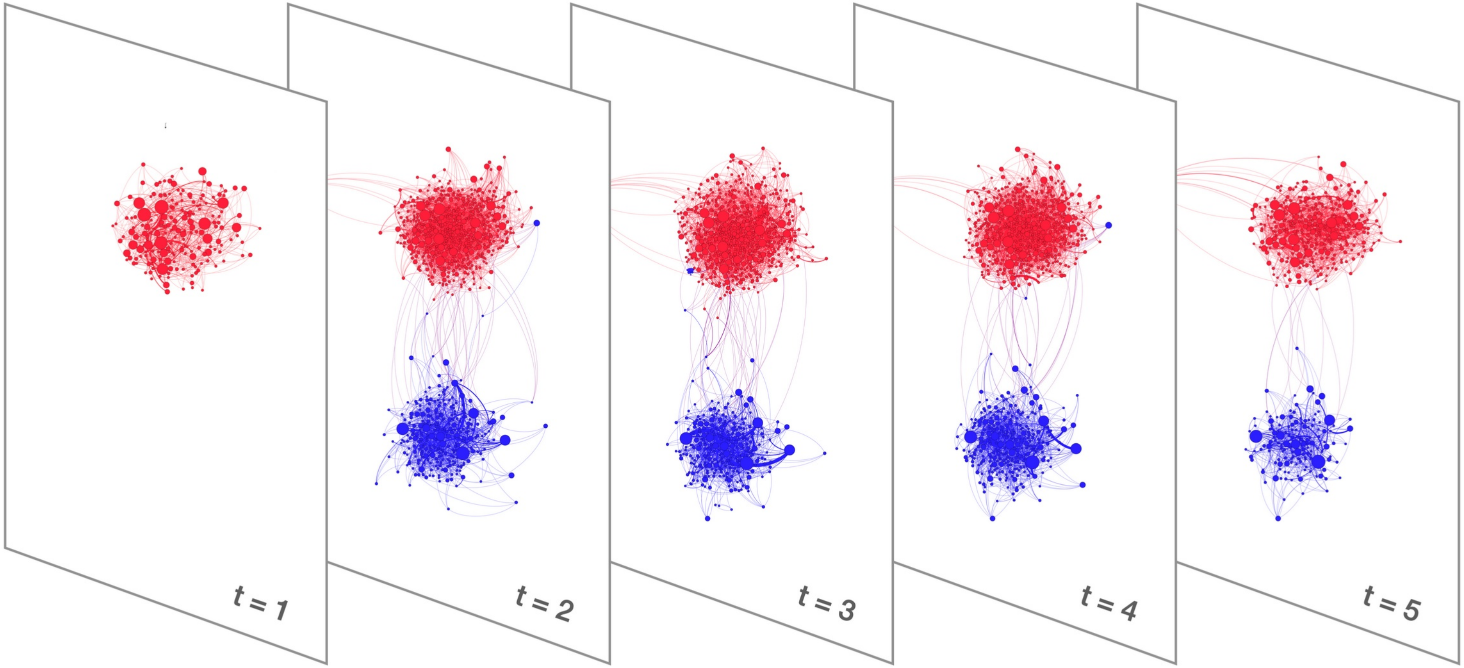
(a) Work. (b) Lunch. (c) Facebook. (d) Friend. (e) Coauthor.

https://dl.acm.org/doi/abs/10.1145/2898362

Some graphs are time-evolving

Temporal, or "time-evolving" graph

At each timestep there are new nodes and/or edges (and/or deletions)



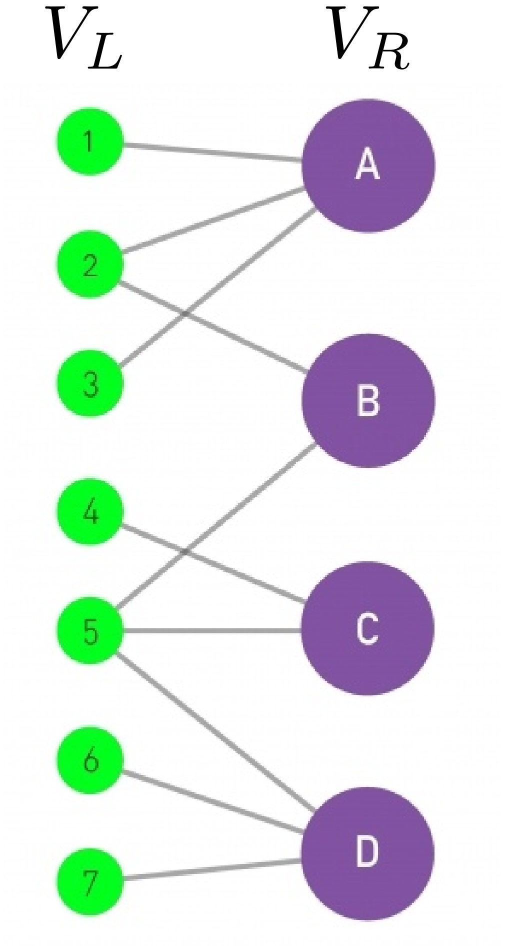
Source: Menczer, Fortunato, David: A First Course on Networks Science. Cambridge, 2020.

Some graphs are bi-partite

- A **bipartite** graph is a graph

  G = (V,E) such that

$$V = V_L \cup V_R, V_L \cap V_R = \emptyset, E \subseteq V_L \times V_R$$

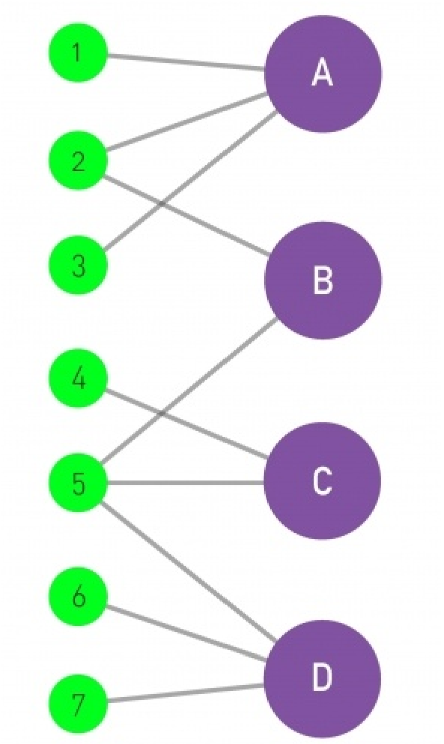


Exercise: project a bipartite network

?

Left projection: graph where nodes are 1, 2, …, 7 and nodes are connected if they share a neighbor

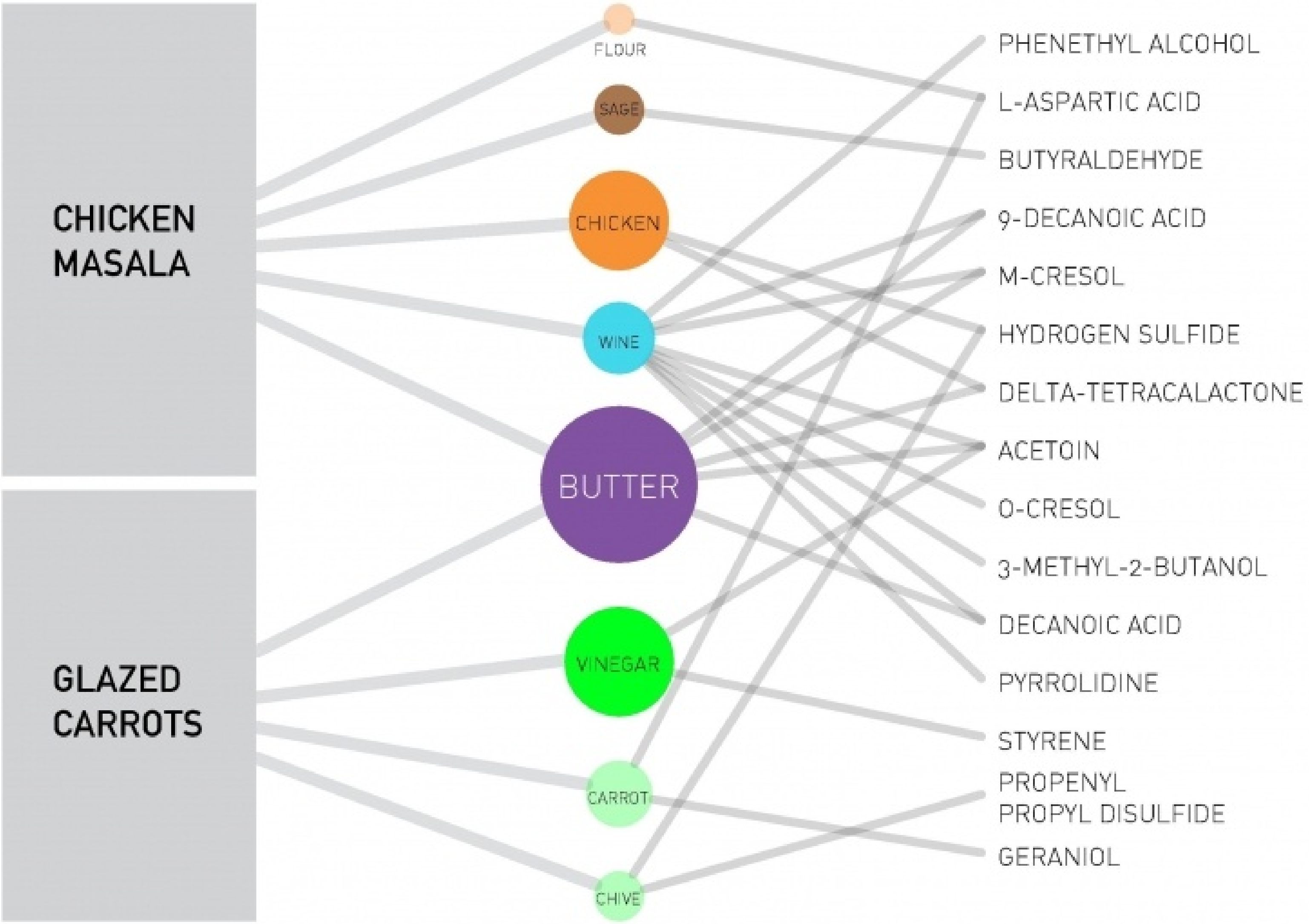
V_L V_R



?

Right projection: graph where nodes are A, B, …, D and nodes are connected if they share a neighbor

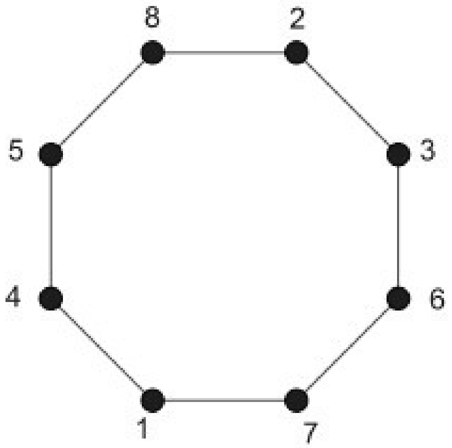
Tripartite network



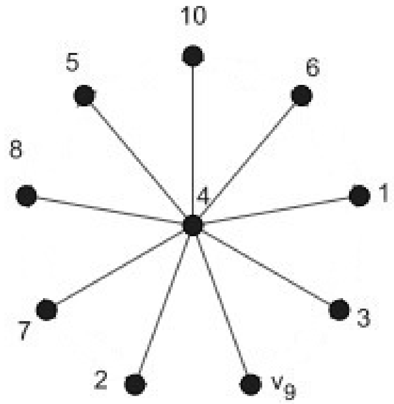
Some graphs have a name

Line

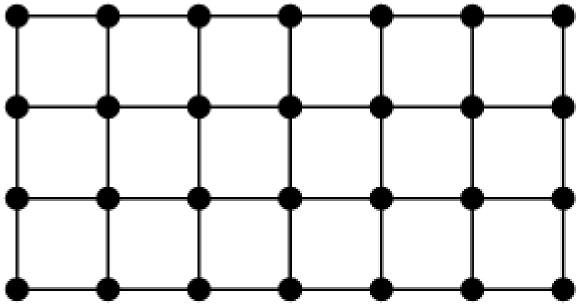
Cycle



Star



Lattice



Clique and Bi-partite clique

- A **clique** is a complete (sub)graph: $E = (V \times V)$
- An **n-clique** is a complete graph of n nodes
- A **bi-partite clique** is such that

$$V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset, E = (V_1 \times V_2)$$

- A **(n_1, n_2)-clique** is a bipartite clique such that

$$|V_1| = n_1, |V_2| = n_2$$

Note: “clique” in popular culture

In some parts of Latin America, a “***clica***” is a close group of friends, or sometimes a group within a gang

Photo credit: @astro_jr

Summary

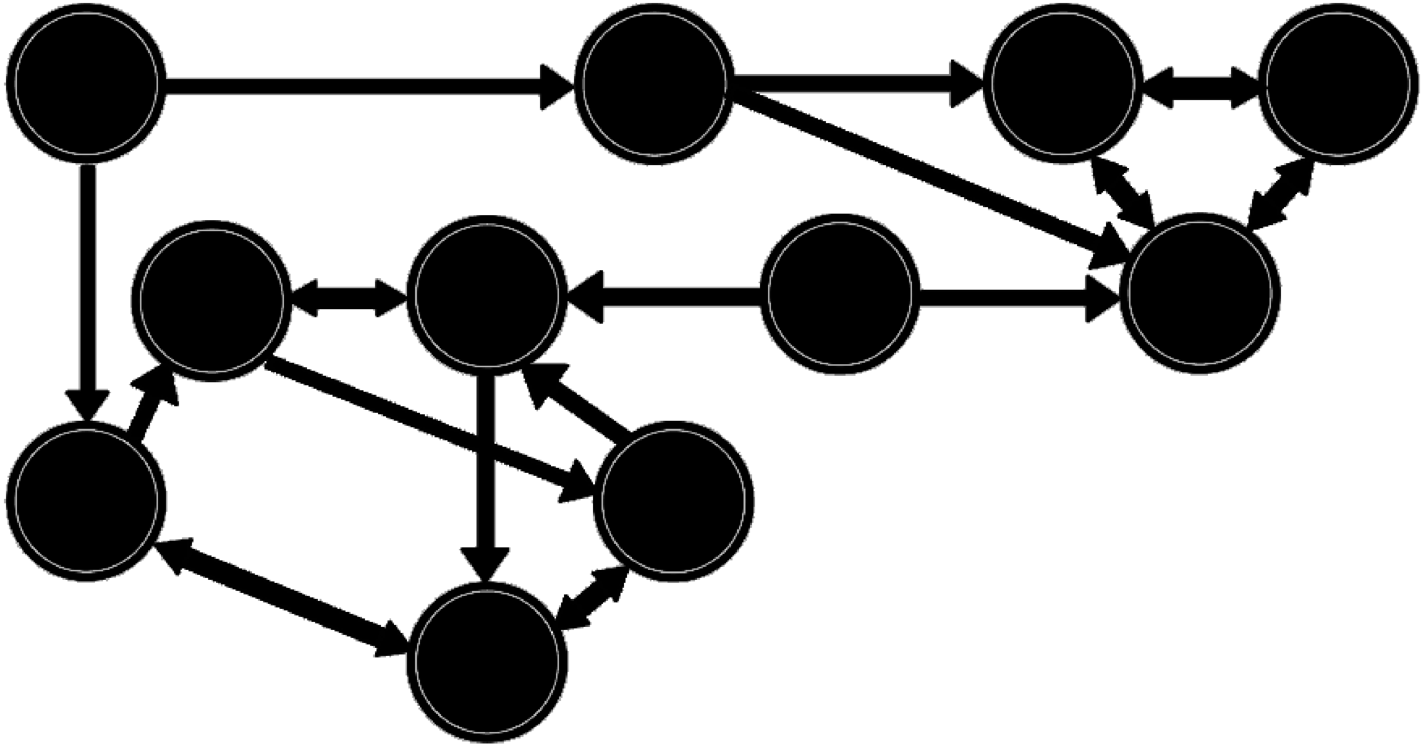
Things to remember

- Definitions
  - degree, in-degree, out-degree
  - time-evolving graph, multi-layer graph
  - line graph, cycle graph, star graph, lattice, bi-partite graph, clique
- Writing the adjacency matrix of a graph, and drawing a graph given its adjacency matrix
- Plotting the degree distribution of a graph
- Projecting a bi-partite graph

Practice on your own

Draw the indegree, outdegree, degree distribution

Write the adjacency matrix



https://www.6seconds.org/2017/07/03/20125/

Practice on your own

How do you call the sub-graph induced by nodesets:

- {H, A, B}
- {G, H, D}
- {B, D, E, G}
- {A, B, D, E}